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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 49, 88–123 (1975)

Invariant Subspaces of Analytic Multiparticle Hamiltonians

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Submitted by C. L. Dolph

The quantum mechanics of n particles interacting through analytic two-body interactions can be formulated as a problem of functional analysis on a Hilbert space $\mathfrak{G}$ consisting of analytic functions. On $\mathfrak{G}$, there is an Hamiltonian H with resolvent $R(\lambda)$. These quantities are associated with families of operators $H(\varphi)$ and $R(\lambda, \varphi)$ on $\mathfrak{L}^2$, the case $\varphi = 0$ corresponding to standard quantum mechanics. The spectrum of $H(\varphi)$ consists of possible isolated points, plus a number of half-lines starting at the thresholds of scattering channels and making an angle 2φ with the real axis.

Assuming that the two-body interactions are in the Schmidt class on the two-particle space $\mathfrak{G}$, this paper studies the resolvent $R(\lambda, \varphi)$ in the case $\varphi \neq 0$. It is shown that a well known Fredholm equation for $R(\lambda, \varphi)$ can be solved by the Neumann series whenever $|\lambda|$ is sufficiently large and λ is not on a singular half-line. Owing to this, $R(\lambda, \varphi)$ can be integrated around the various half-lines to yield bounded idempotent operators $P_p(\varphi)$ ($p = 1, 2,...$) on $\mathfrak{L}^2$. The range of $P_p(\varphi)$ is an invariant subspace of $H(\varphi)$. As φ varies, the family of operators $P_p(\varphi)$ generates a bounded idempotent operator P_p on a space $\mathfrak{G}$. The range of this is an invariant subspace of H. The relevance of this result to the problem of asymptotic completeness is indicated.

1. INTRODUCTION

This is a continuation of earlier papers [1–3], in which the quantum mechanics of n particles was formulated as a problem of functional analysis on a Hilbert space $\mathfrak{G}$ whose elements are analytic functions of complex dynamical variables. The space $\mathfrak{G}$ for n particles consists of functions $f(ke^{i\varphi}, \omega)$ depending on a complex momentum $ke^{i\varphi}$ and on $3n - 4$ real polar angles ω. In order that f be in $\mathfrak{G}$, it must be analytic in $ke^{i\varphi}$ for almost every ω, regular in a sector $\alpha < \varphi < \beta$, and such that the integral

$$\int d\omega \int_0^\infty |f(ke^{i\varphi}, \omega)|^2 k^{3n-4}\, dk \tag{1.1}$$

* This work was supported in part by the National Science Foundation under Grant No. GP-29391.

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exists and is bounded, uniformly in φ for $\alpha < \varphi < \beta$. In the notation of [2, Definition 2.2] the class of all functions satisfying these requirements is denoted by $\mathfrak{G}(\alpha, \beta, 3n-3)$, $\mathfrak{G}(\alpha, \beta)$, or simply $\mathfrak{G}$.

Papers [2] and [3] are devoted to n-particle systems with two-body interactions in the space $\mathfrak{G}(-\gamma, \gamma, 3)$, with some $\gamma > 0$. Such systems give rise to an Hamiltonian H on $\mathfrak{G}$ whose resolvent $R(\lambda)$ can be evaluated as the solution of a Fredholm equation. It was shown in [2] and [3] that H and $R(\lambda)$ are associated with families of operators $H(\varphi)$ and $R(\lambda, \varphi)$ on $\mathfrak{L}^2$. Specifically, if f is in $\mathfrak{G}$, its restriction to fixed φ is in $\mathfrak{L}^2$, by Eq. (1.1). The operator $R(\lambda, \varphi)$ acts on this restriction according to

$$R(\lambda, \varphi) f(ke^{i\varphi}, \omega) = R(\lambda) f(ke^{i\varphi}, \omega), \tag{1.2}$$

the right side of Eq. (1.2) being the restriction of $R(\lambda)f$ to fixed φ. The case $\varphi = 0$ yields standard quantum mechanics, which is thus incorporated in the present formalism.

Of particular interest is the spectrum of $H(\varphi)$. There may be a point spectrum. The location of this does not depend on φ. It corresponds to bound states and resonances. There is certainly a continuous spectrum, consisting of a set of half-lines $Y(\lambda_p, \varphi)$,

$$\lambda = \lambda_p + le^{2i\varphi} \qquad (0 \leqslant l < \infty), \tag{1.3}$$

starting at the thresholds λ_p of scattering channels. This is explained in [3, Theorems 6.20 and 6.21]. The half-line $Y(0, \varphi)$ is included in the set. If $\lambda_p \neq 0$, then the notation of [3] says that

$$\lambda_p = \sum_{i=1}^{j} \lambda_{q_i}^{(n_i)}. \tag{1.4}$$

This decomposition corresponds to a division of the n-particle system into l disjoint groups of n_i particles ($i = 1,..., l, \sum_{i=1}^{l} n_i = n$). In the case of Eq. (1.4), $n_i \geqslant 2$ for $i = 1,..., j$, so $n_{j+1} = \cdots = n_l = 1$. The number $\lambda_{q_i}^{(n_i)}$ is an eigenvalue of the Hamiltonian $H^{(n_i)}(\varphi)$ of group i. This strongly suggests that the half-line $Y(\lambda_p, \varphi)$ refers to the l groups being scattered at one another, groups $1,..., j$ being bound in eigenstates of their Hamiltonians having eigenvalues $\lambda_{q_i}^{(n_i)}$. It is convenient to refer to this process as scattering in channel p.

If $\varphi = 0$, the half-lines $Y(\lambda_p, \varphi)$ all coincide with the real axis. We now refer to standard quantum mechanics. For a large class of interactions, it is well known [4, 5] that each channel p is associated with wave operators Ω_{+p} and Ω_{-p}. These are partial isometries on $\mathfrak{L}^2$. The operators $\Omega_{+p}\Omega_{+p}^*$

and $\Omega_{-p}\Omega^*_{-p}$ are orthogonal projections. Their ranges are invariant subspaces of $H(0)$. The ranges are mutually orthogonal in the sense that

$$\Omega^*_{+p}\Omega_{+q} = 0, \qquad \Omega^*_{-p}\Omega_{-q} = 0 \qquad (p \neq q). \tag{1.5}$$

If a scattering system is in channel p as the time t tends to $-\infty$, then its wave function is in the range of Ω_{+p} at all times. If the system is in channel p as the time t tends to ∞, then the wave function is in the range of Ω_{-p} at all times.

Let the projection onto the space spanned by the eigenstates of $H(0)$ be denoted by B. Then

$$B\Omega_{+p} = B\Omega_{-p} = 0. \tag{1.6}$$

There is a long-standing conjecture that

$$B + \Sigma_p \Omega_{+p}\Omega^*_{+p} = B + \Sigma_p \Omega_{-p}\Omega^*_{-p} = I, \tag{1.7}$$

I being the identity operator. If Eq. (1.7) is satisfied, the system is called asymptotically complete. It is an open problem whether asymptotic completeness holds true in general n-particle systems.

The present paper considers the case $\varphi \neq 0$. Choosing f and g in $\mathfrak{L}^2$, we study the integral

$$(2\pi i)^{-1} \int_{C_p} (R(\lambda, \varphi) f, g)\, d\lambda \tag{1.8}$$

taken along a contour C_p running from $\infty e^{2i\varphi}$ to $\infty e^{2i\varphi}$ in such a way that $\Upsilon(\lambda_p, \varphi)$ is to the right of C_p. The most interesting case arises if the rest of the spectrum of $H(\varphi)$ is to the left of C_p, so that C_p runs between branch cuts of $R(\lambda, \varphi)$, but this is not implied by the notation.

It is a major problem to show that the integral (1.8) actually converges, the difficulty being that $H(\varphi)$ is not normal. Hence, there are no general theorems available concerning the behavior of the norm $\| R(\lambda, \varphi)\|$ as a function of λ. It appears that relevant information can only be obtained by explicit evaluation. To this end, we use the Fredholm equation for $R(\lambda, \varphi)$ that was first proposed by Weinberg [6] and the author [7], investigating what happens to the solution if λ tends to $\infty e^{2i\varphi}$.

We have not succeeded in finding the asymptotic behavior of $R(\lambda, \varphi)$ in the case of the local two-body interactions considered in [2] and [3]. The present paper applies to two-body interactions in the class $\mathfrak{K}$ on the two-particle space $\mathfrak{G}(-\gamma, \gamma, 3)$, with $\gamma > 0$. This point is explained in Sec. 2. The class $\mathfrak{K}$ is a subclass of the Schmidt class on $\mathfrak{G}$. It is easy to see that papers [2] and [3] remain essentially true for the present class of interactions, proofs only requiring occasional minor changes.

It is shown in Section 3 how the integral (1.8) can be treated in the case of two particles. The integrand is examined on the lines

$$\begin{aligned} \lambda &= \lambda_p + (l - i\epsilon)\, e^{2i\varphi} \qquad (-\infty < l < \infty), \\ \lambda &= \lambda_p + (l + i\epsilon)\, e^{2i\varphi} \qquad (-\infty < l < \infty), \end{aligned} \tag{1.9}$$

it being understood that λ_p can only take the value 0 if $n = 2$. It is shown that the Fredholm equation for $R(\lambda, \varphi)$ can be solved by the Neumann series provided l is sufficiently large. From this it is first deduced that $\| R(\lambda, \varphi) f \|$ is a square-integrable function of l. The expression (1.8) is then given a meaning as a principal-value integral.

Once it has been established that the integral (1.8) exists in a suitable sense, it is easy to show that it defines a bounded idempotent operator $P_p(\varphi)$ on $\mathfrak{L}^2$. This is discussed in Section 8, Sections 4–7 being required to prepare for the integral (1.8) in the case of n particles.

In treating general numbers of particles, we find it convenient to look at the Fourier transform of $R(\lambda, \varphi)$ with respect to l, the variables l and λ being related according to either expression (1.9). By way of introduction, Section 4 is devoted to general lemmas on l-dependent operators $A(l)$ on $\mathfrak{L}^2$ and their Fourier transforms $A^\wedge(t)$, with special emphasis on the Schmidt class and on convolutions.

Writing the n-particle resolvent equation in the form

$$R^{(n)}(\lambda, \varphi) = Q^{(n)}(\lambda, \varphi) + K^{(n)}(\lambda, \varphi)\, R^{(n)}(\lambda, \varphi), \tag{1.10}$$

we discuss the inhomogeneous term $Q^{(n)}$ and its Fourier transform $Q^{(n)\wedge}$ in Section 5. Under suitable assumptions on $R^{(m)}$ for $m = 2, 3, \ldots, n - 1$, it is found that $\| Q^{(n)}(\lambda, \varphi) f \|$ is bounded and square-integrable with respect to l, that $\| Q^{(n)\wedge}(t) \|$ is bounded and integrable. Under the same assumptions on $R^{(m)}$, Sec. 6 shows that the Schmidt norm $\sigma[K^{(n)}(\lambda, \varphi)]$ is bounded and square-integrable with respect to l, that $\sigma[K^{(n)\wedge}(t)]$ is bounded and integrable. Furthermore, $\sigma[K^{(n)}(\lambda, \varphi)]$ tends to 0 as l tends to ∞, this corresponding to λ tending to $\infty e^{2i\varphi}$ along a line (1.9) parallel to, but not coinciding with, a branch cut of $R^{(n)}(\lambda, \varphi)$.

Under the assumptions of Sections 5 and 6, the Fredholm equation (1.10) can be solved by the Neumann series if l is sufficiently large. This result is used in an iteration argument in Sec. 7 to show that $Q^{(n)}$, $K^{(n)}$, and their Fourier transforms do indeed have the properties quoted above, for all n. In addition, $\| R^{(n)}(\lambda, \varphi) f \|$ is bounded and square-integrable with respect to l, while $\| R^{(n)\wedge}(t) f \|$ is bounded.

It should be emphasized that the above statements are all subject to the condition that $\epsilon \neq 0$ in Eq. (1.9). Even so, they suffice to define bounded idempotent operators $P_p(\varphi)$ with the help of Eq. (1.8). If $\lambda_p \neq \lambda_q$ and it is understood that $Y(\lambda_p, \varphi)$ is to the right of C_p but to the left of C_q, with $Y(\lambda_q, \varphi)$ to the right of C_q but to the left to C_p, then

$$P_p(\varphi)\, P_q(\varphi) = P_q(\varphi)\, P_p(\varphi) = 0. \tag{1.11}$$

Further properties of $P_p(\varphi)$ are listed in Section 8. In particular, it is shown that $P_p(\varphi)$ commutes with $R(\lambda, \varphi)$. Hence, its range is an invariant subspace of $H(\varphi)$.

In Section 9, the angle φ is varied, with the result that the half-line $Y(\lambda_p, \varphi)$ rotates about λ_p. In doing so, it may pass through poles or branch points of $R(\lambda, \varphi)$. Suppose, however, that α and β are such that $Y(\lambda_p, \varphi)$ does not pass through any poles or branch points, other than λ_p, as φ runs from α to β. This assumption implies that either $0 < \alpha < \beta < \pi/2$, or $-\pi/2 < \alpha < \beta < 0$. If it is satisfied, then $P_p(\varphi) f$ belongs to $\mathfrak{G}(\alpha, \beta)$ whenever f belongs to $\mathfrak{G}(\alpha, \beta)$. If f and g are in $\mathfrak{G}(\alpha, \beta)$, then the integral

$$\int d\omega \int_0^\infty g(ke^{i\varphi}, \omega)\, P_p(\varphi) f(ke^{i\varphi}, \omega)\, (ke^{i\varphi})^{3n-4}\, e^{i\varphi}\, dk \tag{1.12}$$

exists for $\alpha \leqslant \varphi \leqslant \beta$ and does not depend on φ. This is shown in Section 9. It follows that there is a bounded idempotent operator P_p on $\mathfrak{G}(\alpha, \beta)$ whose range is an invariant subspace of H.

It may be helpful to indicate how the above results relate to the problem of asymptotic completeness. Details are to be presented in separate papers. If the number of poles and branch points is finite, so that they do not cluster around $\lambda = \lambda_p$, one can choose $\beta > 0$ but so close to 0 that there are no poles or branch points between $Y(\lambda_p, \varphi)$ and the positive real axis. If one now takes any small α such that $0 < \alpha < \beta$, this yields an interval $\alpha \leqslant \varphi \leqslant \beta$ in which the integral (1.12) does not depend on φ. At this stage, there is no justification for letting α tend to 0. Even so, one may hope that there is enough continuity in the formalism to show that the integral (1.12) is, in fact, equal to an expression of the form

$$\int d\omega \int_0^\infty g(k, \omega)\, P_{+p}(0) f(k, \omega)\, k^{3n-4}\, dk, \tag{1.13}$$

with some bounded operator $P_{+p}(0)$ on $\mathfrak{L}^2$. Similarly, by letting φ tend to 0 through negative values, one may hope to define an operator $P_{-p}(0)$. Now suppose that the contour C_p that determines $P_p(\varphi)$, separates $Y(\lambda_p, \varphi)$ from the rest of the spectrum of $H(\varphi)$. The obvious conjecture is then that the two operators $P_p(0)$ are precisely the two projections $\Omega_p \Omega_p{}^*$.

Again, let the number of poles and branch points be finite and let φ be so close to 0 that there are no poles or branch points between $Y(\lambda_p, \varphi)$ and the positive real axis. Let C_p separate $Y(\lambda_p, \varphi)$ from the rest of the spectrum of $H(\varphi)$. Let $B(\varphi)$ denote the projection onto the subspace spanned by the eigenstates of $H(\varphi)$. Then it is easy to see that

$$B(\varphi) + \Sigma_p P_p(\varphi) = I. \tag{1.14}$$

If φ tends to 0, there is no difficulty in showing that $B(\varphi)$ tends to B. One may therefore hope that Eq. (1.14) tends to the desired relation (1.7).

Because of our earlier assumption, the above applies only to two-body interactions in the Schmidt class. Once asymptotic completeness has been established for this case, one may hope to go over to local interactions with the help of limiting procedures. This point will also be worked out in a separate paper.

We conclude this introduction with references to related investigations. For $n = 2$, asymptotic completeness was proved by Kuroda [8, 9]. He first considered interactions in the trace class, then extended his result to local interactions by a limiting procedure. Other proofs for $n = 2$ and local interactions are due to Ikebe [10] and Faddeev [11], Faddeev's work also covering the case $n = 3$. A proof for $n = 4$ was given by Hepp [12]. In case $n > 4$, Hepp's paper shows that there is asymptotic completeness in systems in which the interaction is either repulsive, or else so weak that the Born series converges. Further results for weak and repulsive interactions are due to Iorio and O'Carroll [13] and Lavine [14], respectively. These cases obviously exclude the possibility of bound fragments being scattered. In genuine multi-channel systems of arbitrary n, there is asymptotic completeness in the subspace in which there is not enough energy available to break the system up into four or more fragments. This way shown by Schtalheim [15].

It appears that the idea of considering analytic interactions and complex dynamical variables goes back to Bottino, Longoni, and Regge [16]. A recent application most closely related to part of our work, is due to Aguilar and Combes [17], and Balslev and Combes [18]. These authors assume two-body interactions that are analytic with respect to the dilatation group, their class of allowed interactions being larger than ours. Although there is a difference in language, there is an obvious overlap between their work and our papers [2] and [3]. Either approach yields operators $H(\varphi)$, the emphasis in the Aguilar-Balslev–Combes theory being on the spectral properties of these. In their introduction, there is a reference to the problem of asymptotic completeness, but it appears that their method has not yet been applied to this. Neither has it been used to identify invariant subspaces of $H(\varphi)$.

2. The Interaction

The interaction V is a sum of two-body terms V_{jl}. The present paper assumes that each V_{jl} belongs to the class $\mathfrak{K}$ on the space $\mathfrak{G}(-\gamma, \gamma, 3)$, with some $\gamma > 0$. Thus, V_{jl} is an integral operator whose kernel $V_{jl}(k, \omega, k', \omega', \varphi)$ has the property that

$$\int d\omega\, d\omega' \int_0^\infty | V_{jl}(k, \omega, k', \omega', \varphi)|^2 k^2 k'^2\, dk\, dk' \tag{2.1}$$

exists and is bounded, uniformly in φ for $-\gamma < \varphi < \gamma$. In order that V_{jl} be in $\mathfrak{K}$, its kernel must also satisfy an analyticity requirement. To be specific, it must depend only on $(k^2 + k'^2)^{1/2} e^{i\varphi}$, k/k', ω, and ω', being analytic in $(k^2 + k'^2)^{1/2} e^{i\varphi}$ for almost every k/k', ω, ω'. This follows from [1, Theorem 6.2]. Alternative characterizations of the class $\mathfrak{K}$ are provided by [1, Definition 3.1] and [1, Theorem 3.5].

If

$$-\gamma \leqslant \alpha < \beta \leqslant \gamma \tag{2.2}$$

and f belongs to $\mathfrak{G}(\alpha, \beta, 3)$, then [1, Eq. (3.20)] says that

$$V_{jl} f(ke^{i\varphi}, \omega) = \int d\omega' \int_0^\infty V_{jl}(k, \omega, k', \omega', \varphi) f(k' e^{i\varphi}, \omega')\, (k' e^{i\varphi})^2 e^{i\varphi}\, dk'. \tag{2.3}$$

The class $\mathfrak{K}$ is a proper subclass of the Schmidt class on $\mathfrak{G}$ by [1, Theorem 3.11]. If Eq. (2.3) is restricted to some fixed φ, it yields an operator in the Schmidt class on $\mathfrak{L}^2$. This we denote by $V_{jl}(\varphi)$, as in Eq. (1.2). The set of all operators $V_{jl}(\varphi)$ obtained in this way from operators in $\mathfrak{K}$ is dense, in the Schmidt norm, in the Schmidt class on $\mathfrak{L}^2$. This can be deduced from [1, Theorems 2.17 and 6.2].

In order that $H(0)$ be self-adjoint, we assume that

$$\overline{V}_{jl}(k', \omega', k, \omega, 0) = V_{jl}(k, \omega, k', \omega', 0). \tag{2.4}$$

This equation is of the same form as [2, Eq. (4.14)]. It follows that

$$\overline{V}_{jl}(k', \omega', k, \omega, -\varphi) = V_{jl}(k, \omega, k', \omega', \varphi). \tag{2.5}$$

It is obvious from the foregoing how V_{jl} and $V_{jl}(\varphi)$ act on functions in $\mathfrak{G}(\alpha, \beta, 3n - 3)$ and $\mathfrak{L}^2(3n - 3)$, respectively. As in [2] and [3], we write

$$H(\varphi) = H_0(\varphi) + V(\varphi) = k^2 e^{2i\varphi} + \sum_{j<l} V_{jl}(\varphi). \tag{2.6}$$

The domain $\mathfrak{D}[H(\varphi)]$ of $H(\varphi)$ is the set of all $f(k, \omega)$ in $\mathfrak{L}^2$ having the property that $k^2 f(k, \omega)$ is in $\mathfrak{L}^2$. Thus, the domain does not depend on φ. With Eq. (2.5), it follows that

$$H^*(-\varphi) = H(\varphi). \tag{2.7}$$

3. Two Particles

To introduce the way in which one can define the integral (1.8), we first discuss the case of two particles. Writing

$$R_0(\lambda, \varphi) = (k^2 e^{2i\varphi} - \lambda)^{-1}, \qquad K(\lambda, \varphi) = -R_0(\lambda, \varphi)\, V(\varphi), \tag{3.1}$$

and denoting the Schmidt norm of $K(\lambda, \varphi)$ by $\sigma[K(\lambda, \varphi)]$, we need the resolvent equation

$$R(\lambda, \varphi) = R_0(\lambda, \varphi) + K(\lambda, \varphi)\, R(\lambda, \varphi). \tag{3.2}$$

Lemma 3.1. *Let $f(k, \omega)$ belong to $\mathfrak{L}^2(3)$, choose some fixed $\zeta > 0$, and allow ϵ to take values in the intervals $[\zeta, \infty)$ and $(-\infty, -\zeta]$. Write*

$$\lambda = (l + i\epsilon)\, e^{2i\varphi} \qquad (-\infty < l < \infty). \tag{3.3}$$

If $V(\varphi)$ is as in Sec. 2, then $R_0(\lambda, \varphi)$ and $K(\lambda, \varphi)$ have the following properties.

(1) *The norm $\| R_0(\lambda, \varphi)\|$ is bounded uniformly in λ.*

(2) *There exists a constant c such that*

$$\int_{-\infty}^{\infty} \| R_0(\lambda, \varphi) f \|^2 \, dl < c \, \| f \|^2, \tag{3.4}$$

for all f in $\mathfrak{L}^2$, uniformly in ϵ.

(3) *The kernel $K(\lambda, \varphi)$ belongs to the Schmidt class, its Schmidt norm being bounded uniformly in λ.*

(4) *There exists a constant d such that*

$$\int_{-\infty}^{\infty} \{\sigma[K(\lambda, \varphi)]\}^2 \, dl < d, \tag{3.5}$$

uniformly in ϵ.

(5) *The integrand in Eq. (3.5) tends to 0 as l tends to $\pm\infty$, uniformly in ϵ.*

Proof. Properties (1)–(4) can easily be checked by direct evaluation. As for property (5), given $V(\varphi)$, ζ, and some $\delta > 0$, we first choose L so large that

$$\int d\omega\, d\omega' \int_0^\infty k'^2\, dk' \int_L^\infty |\, V(k, \omega, k', \omega', \varphi)|^2\, k^2\, dk < \delta\zeta^2. \tag{3.6}$$

Next, l is chosen so large that $l > L^2$ and

$$\int d\omega\, d\omega' \int_0^\infty |\, V(k, \omega, k', \omega', \varphi)|^2\, k^2 k'^2\, dk\, dk' < \delta[(L^2 - l)^2 + \zeta^2]. \tag{3.7}$$

By Eq. (3.6), the integral

$$\int d\omega\, d\omega' \int_0^\infty [(k^2 - l)^2 + \epsilon^2]^{-1} \,|\, V(k, \omega, k', \omega', \varphi)|^2\, k^2 k'^2\, dk\, dk' \tag{3.8}$$

now receives a contribution from the region $k^2 > L^2$ which is less than δ. By Eq. (3.7), the contribution from the region $k^2 < L^2$ is also less than δ. Hence, the integral tends to 0 as l tends to ∞. It is obvious that it also tends to 0 as l tends to $-\infty$. This proves the lemma.

COROLLARY 3.2. *If* $|\, l \,|$ *is sufficiently large, then* $R(\lambda, \varphi)$ *can be evaluated with the help of the Born series.*

DEFINITION 3.3. For any $\zeta > 0$, the symbol $\Gamma(\zeta, \varphi)$ stands for the region

$$\operatorname{dist}[\lambda - \text{spectrum } H(\varphi)] \geqslant \zeta > 0. \tag{3.9}$$

The expression "uniform in λ" means "uniform in λ, provided ζ is held fixed and λ is in $\Gamma(\zeta, \varphi)$."

LEMMA 3.4. *Let the data be as in Lemma* 3.1. *Then* $R(\lambda, \varphi)$ *has the following properties.*

(1) *The norm* $\| R(\lambda, \varphi)\|$ *is bounded uniformly in* λ.

(2) *There exists a constant c such that*

$$\int_{-\infty}^{\infty} \| R(\lambda, \varphi) f \|^2\, dl < c \,\| f \|^2 \tag{3.10}$$

whenever the line (3.3) *is in* $\Gamma(\zeta, \varphi)$, *for all f in* $\mathfrak{L}^2$.

Proof. For any $L > 0$, let $R_L(\lambda, \varphi)$ be defined by

$$\begin{aligned} R_L(\lambda, \varphi) &= R(\lambda, \varphi) \qquad (|\,\lambda\,| < L), \\ &= 0 \qquad\qquad\; (|\,\lambda\,| > L), \end{aligned} \tag{3.11}$$

and similarly for $R_{0L}(\lambda, \varphi)$. It is clear that $R_L(\lambda, \varphi)$ has the properties (1) and (2).

Given any $\delta > 0$, Lemma 3.1 allows us to make L so large that $\| K(\lambda, \varphi)\| < \delta$ if $|\lambda| > L$ and λ is in $\Gamma(\zeta, \varphi)$. This gives

$$\begin{aligned} \|[R(\lambda, \varphi) - R_L(\lambda, \varphi)] f\| &= \|[1 - K(\lambda, \varphi)]^{-1} [R_0(\lambda, \varphi) - R_{0L}(\lambda, \varphi)] f\| \\ &\leqslant (1-\delta)^{-1} \| R_0(\lambda, \varphi) f\| . \end{aligned} \tag{3.12}$$

The lemma now follows from properties (1) and (2) of Lemma 3.1.

Remark 3.5. It follows from property (1) of Lemma 3.4 that $\| R(\lambda, \varphi) f\|$ tends to 0 as $|\lambda|$ tends to ∞, uniformly in λ. To see this, we first approximate f by a function g in the domain of $H(\varphi)$. Next, we observe that

$$R(\lambda, \varphi) g = -\lambda^{-1} g + \lambda^{-1} R(\lambda, \varphi) H(\varphi) g. \tag{3.13}$$

This shows that $\| R(\lambda, \varphi) g\|$ tends to 0 as $|\lambda|$ tends to ∞, uniformly in λ. Hence, so does $\| R(\lambda, \varphi) f\|$.

With a view to the integral (1.8), we now choose $\epsilon > 0$ and write

$$\lambda_+ = (l + i\epsilon) e^{2i\varphi}, \qquad \lambda_- = (l - i\epsilon) e^{2i\varphi} \qquad (-\infty < l < \infty). \tag{3.14}$$

This gives

$$|([R(\lambda_+, \varphi) - R(\lambda_-, \varphi)] f, g)| = 2\epsilon \, |(R(\lambda_+, \varphi) f, R^*(\lambda_-, \varphi) g)| . \tag{3.15}$$

From Eq. (2.7) and the relation

$$[H^*(\varphi) - \bar{\lambda}]^{-1} = [H(\varphi) - \lambda]^{-1*}, \tag{3.16}$$

it follows that

$$R^*(\lambda_-, \varphi) = R([l + i\epsilon] e^{-2i\varphi}, -\varphi). \tag{3.17}$$

Since Lemma 3.4 remains true if φ is replaced by $-\varphi$, we may conclude that either side of Eq. (3.15) is an integrable function of l.

Now let C be the oriented contour consisting of the lines

$$\begin{aligned} \lambda &= (l - i\epsilon) e^{2i\varphi} \qquad (\infty > l > -\infty), \\ \lambda &= (l + i\epsilon) e^{2i\varphi} \qquad (-\infty < l < \infty), \end{aligned} \tag{3.18}$$

it being understood that C is in $\Gamma(\zeta, \varphi)$. Let C_L be the part of C on which $|l| < L$. Then

$$\int_{C_L} (R(\lambda, \varphi) f, g) \, d\lambda = e^{2i\varphi} \int_{-L}^{L} ([R(\lambda_+, \varphi) - R(\lambda_-, \varphi)] f, g) \, dl. \tag{3.19}$$

By the previous paragraph, this tends to a limit as L tends to ∞. We may therefore define

$$\int_C (R(\lambda, \varphi) f, g)\, d\lambda = \lim_{L\to\infty} \int_{C_L} (R(\lambda, \varphi) f, g)\, d\lambda. \tag{3.20}$$

By Eq. (3.15) and property (2) of Lemma 3.4, this yields a quantity that does not exceed some constant times $\| f \| \, \| g \|$. It therefore determines a bounded linear operator on $\mathfrak{L}^2$, as required.

Since $R(\lambda, \varphi)$ is regular on the half-line $\lambda = le^{2i\varphi}$, $l < 0$, the contour C may be replaced by the part on which $l \geqslant 0$, plus some bounded curve from $-i\epsilon e^{2i\varphi}$ to $i\epsilon e^{2i\varphi}$. The integral (3.20) does not depend on ϵ due to Remark 3.5.

In the case of two particles, $R(\lambda, \varphi)$ has only one branch cut. Hence, we could define a projection operator by starting from the identity and subtracting the projection onto the space spanned by the eigenstates from it. This, however, is a procedure that is not sufficiently powerful for systems with more branch cuts.

4. Several Lemmas

We proceed to formulate some lemmas that will enable Section 3 to be generalized to larger numbers of particles. Throughout the present section, $A(l)$ and $B(l)$ are linear operators on an $\mathfrak{L}^2$-space of functions $f(r)$. They are defined for almost every real l.

Lemma 4.1. *If there is a constant a such that*

$$\int_{-\infty}^{\infty} \| A(l) f \|^2 \, dl < a \| f \|^2, \tag{4.1}$$

for all f in a domain $\mathfrak{D}$, then the relation

$$A\hat{}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ilt} A(l)\, dl \tag{4.2}$$

defines a linear operator $A\hat{}(t)$ on $\mathfrak{D}$ for almost every real t. It satisfies

$$\int_{-\infty}^{\infty} \| A\hat{}(t) f \|^2 \, dt = \int_{-\infty}^{\infty} \| A(l) f \|^2 \, dl \tag{4.3}$$

and

$$A(l) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ilt} A\hat{}(t)\, dt. \tag{4.4}$$

Proof. By the data, $A(l)f(r)$ is an $\mathfrak{L}^2$-function of l and r. The lemma follows from the general theory of Fourier transforms.

LEMMA 4.2. *If there is a constant a such that*

$$\int_{-\infty}^{\infty} \| A(l)f \| \, dl < a \| f \| \tag{4.5}$$

for all f in a domain $\mathfrak{D}$, *then the relation* (4.2) *defines a linear operator* $A^{\wedge}(t)$ *on* $\mathfrak{D}$ *such that* $\| A^{\wedge}(t) \|$ *is bounded uniformly in t and* $\| A^{\wedge}(t) f \|$ *tends to* 0 *as t tends to* $\pm\infty$.

Proof. From the data it is obvious that

$$\| A^{\wedge}(t) f \|^2 = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ilt+imt} (A(l)f, A(m)f) \, dl \, dm \leqslant (2\pi)^{-1} a^2 \| f \|^2. \tag{4.6}$$

In the second member, $l - m$ may be chosen as one of the integration variables. Hence, the left side is the Fourier transform of an integrable function, which tends to 0 as t tends to $\pm\infty$.

LEMMA 4.3. *If* $A(l)$ *belongs to the Schmidt class and its Schmidt norm is in* $\mathfrak{L}^2$, *then* $A^{\wedge}(t)$ *belongs to the Schmidt class for almost every t and satisfies*

$$\int_{-\infty}^{\infty} \{\sigma[A^{\wedge}(t)]\}^2 \, dt = \int_{-\infty}^{\infty} \{\sigma[A(l)]\}^2 \, dl. \tag{4.7}$$

Proof. The operator $A(l)$ is an integral operator whose kernel $A(r, r', l)$ is an $\mathfrak{L}^2$-function of r, r', and l. Hence, the Fourier transform $A^{\wedge}(r, r', t)$ exists for almost every t, with

$$\int_{-\infty}^{\infty} dt \int | A^{\wedge}(r, r', t)|^2 \, dr \, dr' = \int_{-\infty}^{\infty} dl \int | A(r, r', l)|^2 \, dr \, dr'. \tag{4.8}$$

This is precisely Eq. (4.7).

LEMMA 4.4. *If* $A(l)$ *belongs to the Schmidt class and its Schmidt norm is in* $\mathfrak{L}^1$, *then* $A^{\wedge}(t)$ *belongs to the Schmidt class. Its Schmidt norm is bounded uniformly in t and tends to* 0 *as t tends to* $\pm\infty$.

Proof. This lemma is related to Lemmas 4.2 and 4.3. In fact,

$$
\begin{aligned}
\{\sigma[A^{\wedge}(t)]\}^2 &= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ilt+imt}\, dl\, dm \int A(r, r', l)\, \bar{A}(r, r', m)\, dr\, dr' \\
&\leqslant (2\pi)^{-1} \int_{-\infty}^{\infty} \sigma[A(l)]\, \sigma[A(m)]\, dl\, dm < \infty.
\end{aligned} \tag{4.9}
$$

That $\sigma[A^{\wedge}(t)]$ tends to 0 as t tends $\pm\infty$ can be shown as in the proof of Lemma 4.2.

LEMMA 4.5. *If $A(l)$ and its adjoint $A^*(l)$ are as in Lemma 4.1, then the adjoint $A^{\wedge *}(t)$ of $A^{\wedge}(t)$ exists for almost every t and satisfies*

$$A^{\wedge *}(t) \supseteq A^{*\wedge}(-t). \tag{4.10}$$

Proof. For f in $\mathfrak{D}$, and g in the domain $\mathfrak{D}^*$ of $A^*(l)$, we have for almost every t

$$
\begin{aligned}
(A^{\wedge}(t) f, g)^* &= (2\pi)^{-1/2} \left[\int_{-\infty}^{\infty} e^{-ilt} (A(l) f, g)\, dl\right]^* \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ilt} (A^*(l)\, g, f)\, dl = (A^{*\wedge}(-t)\, g, f).
\end{aligned} \tag{4.11}
$$

In the first and second members, the stars denote complex conjugates. The lemma follows from Eq. (4.11) by the definition of an adjoint.

LEMMA 4.6. *Let the data be as in Lemma 4.5. Let $B(l)$ have the properties of Lemma 4.1 and suppose that the ranges of $B(l)$ and $B^{\wedge}(t)$ are in $\mathfrak{D}$. Then*

$$\int_{-\infty}^{\infty} (A(l)\, B(l) f, g)\, dl = \int_{-\infty}^{\infty} (A^{\wedge}(t)\, B^{\wedge}(-t) f, g)\, dt \tag{4.12}$$

for all f in the domain of $B(l)$ and all g in the domain of $A^*(l)$.

Proof. If f and g are in the respective domains, the data imply that the integral

$$I = \int_{-\infty}^{\infty} (B(l) f, A^*(l)\, g)\, dl \tag{4.13}$$

converges. By the theory of Fourier transforms and Lemma 4.5,

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} (B^{\wedge}(t) f, A^{*\wedge}(t)\, g)\, dt \\
&= \int_{-\infty}^{\infty} (B^{\wedge}(t) f, A^{\wedge *}(-t)\, g)\, dt = \int_{-\infty}^{\infty} (A^{\wedge}(t)\, B^{\wedge}(-t) f, g)\, dt.
\end{aligned} \tag{4.14}
$$

This proves the lemma.

COROLLARY 4.7. *If the data are as in Lemma* 4.6, *then*

$$\int_{-\infty}^{\infty} e^{-ilt}(A(l)\,B(l)\,f, g)\,dl = \int_{\infty}^{\infty} (A^{\wedge}(u)\,B^{\wedge}(t-u)\,f, g)\,du. \tag{4.15}$$

Proof. If $B(l)$ has transform $B^{\wedge}(u)$, then $e^{-ilt}B(l)$ has transform $B^{\wedge}(t+u)$. By Lemma 4.6, the right side of Eq. (4.15) requires the transform of $e^{-ilt}B(l)$ at $-u$.

5. CONVOLUTION INTEGRALS

We are now ready for the inhomogeneous term $Q^{(n)}$ in the resolvent equation (1.10). By [3, Eq. (6.38)], this is a sum of resolvents $R^{(n)}_{p(l)}$ for n-particle systems that actually consist of l noninteracting groups of fewer particles ($l = 2,..., n$). The subscript $p(l)$ runs through all possible ways of making such noninteracting groups. If the resolvents $R^{(n_i)}$ are known for all groups of fewer particles ($n_i < n$), then $Q^{(n)}$ can be constructed by performing convolutions, according to [3, Eq. (6.33)].

Specifically, let $H_a(\varphi)$ and $H_b(\varphi)$ be typical multiparticle Hamiltonians for different groups, so that they act on different variables, and let their resolvents be $R_a(\lambda, \varphi)$ and $R_b(\lambda, \varphi)$. To find $Q^{(n)}$, we require the convolution

$$[R_a * R_b]\,(\lambda, \varphi) = (2\pi i)^{-1} \int_C R_a(\sigma, \varphi)\,R_b(\lambda - \sigma, \varphi)\,d\sigma, \tag{5.1}$$

where C is a contour in the σ-plane such that the singularities of $R_a(\sigma, \varphi)$ are to the right of C, the singularities of $R_b(\lambda - \sigma, \varphi)$ to the left of C. By [3, Theorem 6.13] the convolution (5.1) can be performed for all λ in the complement of the set of half-lines $Y(\lambda_{ai} + \lambda_{bj}, \varphi)$, where λ_{ai} runs through all branch points λ_{ap} and poles λ_{aq} of R_a, and similarly for λ_{bj}.

From now on, it is essential to assume that $0 < |\varphi| < \pi/2$. This makes it possible for λ to be between successive half-lines $Y(\lambda_{ai} + \lambda_{bj}, \varphi)$. The contour C in the σ-plane will then be meandering between the singularities of $R_a(\sigma, \varphi)$ and those of $R_b(\lambda - \sigma, \varphi)$. Now, a typical multiparticle resolvent $R(\lambda, \varphi)$ may have infinitely many poles and branch points, but clustering of singularities can occur only around singular half-lines $Y(\lambda_p, \varphi)$. There is a finite number of half-lines of accumulation at most. This point is worked out in Remark 7.4. Owing to this, a meandering contour C need only make a finite number of turns. Suppose now, that the integrand in Eq. (5.1) tends to 0 sufficiently fast as $|\sigma|$ tends to ∞. The contour C may then be deformed into a finite number of straight lines making an angle 2φ with the real axis and running between the various cuts and poles. To develop this idea, we adopt the following notation.

DEFINITION 5.1. Let μ be real and such that the line

$$\lambda = \mu + le^{2i\varphi} \qquad (-\infty < l < \infty) \tag{5.2}$$

belongs to the region $\Gamma(\zeta, \varphi)$ of Definition 3.3. Then $\mathscr{R}(l)$ stands for the operator $R(\mu + le^{2i\varphi}, \varphi)$. Furthermore,

$$\begin{aligned}[\mathscr{R}_a * \mathscr{R}_b]\,(l) &= (2\pi)^{-1} \int_{-\infty}^{\infty} \mathscr{R}_a(s)\, \mathscr{R}_b(l-s)\, ds \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} R_a(\mu_a + se^{2i\varphi}, \varphi)\, R_b(\mu_b + [l-s]\, e^{2i\varphi}, \varphi)\, ds.\end{aligned} \tag{5.3}$$

LEMMA 5.2. *Let $\mathscr{R}_a(l)$ be bounded for all real l and suppose that there is a constant c_a such that*

$$\int_{-\infty}^{\infty} \| \mathscr{R}_a(l) f \|^2\, dl < c_a \| f \|^2, \qquad \int_{-\infty}^{\infty} \| \mathscr{R}_a{}^*(l) f \|^2\, dl < c_a \| f \|^2, \tag{5.4}$$

for all f in $\mathfrak{L}^2$. Let $\mathscr{R}_a{}^\wedge(t)$ be bounded for all real t. Suppose that $\mathscr{R}_b(l)$ has similar properties. Then the operators $[\mathscr{R}_a * \mathscr{R}_b]\,(l)$ and $[\mathscr{R}_a * \mathscr{R}_b]^\wedge\,(t)$ exist and are bounded. There is a constant c_{ab} such that

$$\int_{-\infty}^{\infty} \|[\mathscr{R}_a * \mathscr{R}_b]\,(l) f \|^2\, dl < c_{ab} \| f \|^2, \tag{5.5}$$

for all f in $\mathfrak{L}^2$.

Proof. The operator $\mathscr{R}_a(l)$ and its adjoint have the properties of the operator $A(l)$ in Lemma 4.5. In fact, $\mathscr{R}_a{}^\wedge(t)$ and its adjoint also have the properties of $A(l)$ in Lemma 4.5. Similarly, $\mathscr{R}_b{}^\wedge(t)$ is as $B(l)$ in Lemma 4.6. Corollary 4.7 therefore says that

$$\int_{-\infty}^{\infty} e^{ilt} (\mathscr{R}_a{}^\wedge(t)\, \mathscr{R}_b{}^\wedge(t) f, g)\, dt = 2\pi([\mathscr{R}_a * \mathscr{R}_b]\,(l) f, g), \tag{5.6}$$

hence

$$[\mathscr{R}_a * \mathscr{R}_b]^\wedge\,(t) = (2\pi)^{-1/2}\, \mathscr{R}_a{}^\wedge(t)\, \mathscr{R}_b{}^\wedge(t). \tag{5.7}$$

From this it is obvious that $[\mathscr{R}_a * \mathscr{R}_b]^\wedge (t)$ is bounded. The relation (5.5) is easily proved with the help of Lemma 4.1. Furthermore,

$$\begin{aligned} |([\mathscr{R}_a * \mathscr{R}_b] (l) f, g)| &\leqslant (2\pi)^{-1} \int_{-\infty}^{\infty} \| \mathscr{R}_b^\wedge(t) f \| \, \| \mathscr{R}_a^{*\wedge}(-t) g \| \, dt \\ &\leqslant (2\pi)^{-1} (c_a c_b)^{1/2} \| f \| \, \| g \| , \end{aligned} \tag{5.8}$$

where we have used Eq. (4.10) to replace $\mathscr{R}_a^{\wedge *}(t)$ by $\mathscr{R}_a^{*\wedge}(-t)$. This shows that $[\mathscr{R}_a * \mathscr{R}_b] (l)$ is bounded and thereby completes the proof.

Remark 5.3. Suppressing μ and using the notation $\mathscr{R}(l)$ is motivated by the fact that we want to make various statements about $R(\lambda, \varphi)$ that depend on l, but not on μ, provided the line (5.2) is in $\Gamma(\zeta, \varphi)$. It will be understood that $\mathscr{R}^\wedge(t)$ depends on μ. Lemma 5.2 is conceived in terms of a fixed pair μ_a, μ_b, but the particular choice of this is immaterial. If μ_a and μ_b are allowed to take all possible values compatible with the respective lines (5.2) being in $\Gamma_a(\zeta, \varphi)$ and $\Gamma_b(\zeta, \varphi)$, then Lemma 5.2 yields results on the convolution (5.1) throughout the region $\Gamma_{ab}(\zeta, \varphi)$ characterized by

$$\operatorname{dist} \Big[\lambda - \bigcup_{i,j} \Upsilon(\lambda_{ai} + \lambda_{bj} , \varphi)\Big] \geqslant \zeta > 0. \tag{5.9}$$

We wish to study the n-particle resolvent $R^{(n)}(\lambda, \varphi)$ in its region of analyticity $\Gamma^{(n)}(\zeta, \varphi)$. By [3, Section 6.4] the term $Q^{(n)}(\lambda, \varphi)$ and the kernel $K^{(n)}(\lambda, \varphi)$ have the same branch cuts as $R^{(n)}(\lambda, \varphi)$. They do not have any other singularities. Their region Γ is denoted by $\Gamma_0^{(n)}(\zeta, \varphi)$.

As in Definition 3.3, "uniform in λ" means "uniform in λ, provided ζ is held fixed and λ is in the appropriate region $\Gamma(\zeta, \varphi)$." Similarly, "uniform in μ" means "uniform in μ, provided ζ is held fixed and the line (5.2) is in $\Gamma(\zeta, \varphi)$." It is essential that ζ is not supposed to be 0.

LEMMA 5.4. *Let the operators $R^{(m)}(\lambda, \varphi)$ $(m = 2, 3,\ldots, n-1)$ have the following properties.*

(1) *The norm $\| R^{(m)}(\lambda, \varphi)\|$ is bounded uniformly in λ.*

(2) *There exists a constant $c^{(m)}$ such that*

$$\begin{aligned} \int_{-\infty}^{\infty} \| \mathscr{R}^{(m)}(l) f \|^2 \, dl &< c^{(m)} \| f \|^2, \\ \int_{-\infty}^{\infty} \| \mathscr{R}^{(m)*}(l) f \|^2 \, dl &< c^{(m)} \| f \|^2, \end{aligned} \tag{5.10}$$

for all in $\mathfrak{L}^2$, uniformly in μ.

(3) *The norm $\| \mathscr{R}^{(m)\wedge}(t)\|$ is bounded uniformly in t and μ.*

Let the convolution ${}_*R(\lambda, \varphi)$ *with region of analyticity* ${}_*\Gamma(\zeta, \varphi)$ *be defined by*

$$* R(\lambda, \varphi) = [R^{(n_1)} * \cdots * R^{(n_j)}] (\lambda, \varphi), \tag{5.11}$$

it being understood that $n_i < n$ $(i = 1,..., j)$.

Then ${}_*R(\lambda, \varphi)$ *has the properties* (1)–(3). *In evaluating* ${}_*R(\lambda, \varphi)$, *the integration contour in each convolution may be chosen to consist of a finite number of parallel lines* (5.2).

Proof. Owing to property (1), $\| R^{(m)}(\lambda, \varphi) f \|$ tends to 0 if λ is in $\Gamma^{(m)}(\zeta, \varphi)$ and $|\lambda|$ tends to ∞. This can be proved in the same way as Remark 3.5. We know from [3] that $R^{(m)}(\lambda, \varphi)$ is regular, and $\| R^{(m)}(\lambda, \varphi)\|$ of order $O(|\lambda|^{-1})$ for large $|\lambda|$, if λ is in a sector of the form

$$\begin{aligned} 2\varphi < \arg(\lambda - \Lambda) &< 2\pi && \text{if} \quad \varphi > 0, \\ 0 < \arg(\lambda - \Lambda) &< 2\pi + 2\varphi && \text{if} \quad \varphi < 0, \end{aligned} \tag{5.12}$$

Λ being the lower bound of the spectrum of $H(0)$. Now consider the contour C used in evaluating $R^{(n_1)} * R^{(n_2)}$. If this is meandering, the foregoing information suffices to deform C into a set of parallel lines in the σ-plane running between the cuts and poles of $R^{(n_1)}(\sigma, \varphi)$ and those of $R^{(n_2)}(\lambda - \sigma, \varphi)$. The number of lines is finite because the cuts and poles of either resolvent may cluster around a finite number of cuts at most.

Once it is known that the convolution may be performed along parallel lines, it follows from Lemma 5.2 that $R^{(n_1)} * R^{(n_2)}$ has properties (1)–(3), the proof of Lemma 5.2 implying uniformity in λ, μ, and t.

By [3, Theorem 6.9], the convolution $R^{(n_1)} * R^{(n_2)}$ is the resolvent of $H^{(n_1)} + H^{(n_2)}$. For large $|\lambda|$ in the appropriate sector (5.12), it follows from [3, Lemma 6.2] that $R^{(n_1)} * R^{(n_2)}$ may be evaluated with the help of the Born series. Hence, it is of order $O(|\lambda|^{-1})$.

Summarizing, $R^{(n_1)} * R^{(n_2)}$ has all the essential properties of $R^{(m)}$. We may therefore proceed to $R^{(n_1)} * R^{(n_2)} * R^{(n_3)}$, the desired result for ${}_*R(\lambda, \varphi)$ following by induction.

Remark 5.5. Since μ in Definition 5.1 is real, Eqs. (2.7) and (3.16) show that

$$R^*(\mu + le^{2i\varphi}, \varphi) = R(\mu + le^{-2i\varphi}, -\varphi). \tag{5.13}$$

Hence, the two equations (5.10) are of the same nature, differing only in the value of φ that is being considered.

LEMMA 5.6. *The operator* $R_0^{(m)}(\lambda, \varphi)$ $(m = 2, 3,...)$ *and its adjoint have* *the properties* (1)–(3) *of Lemma* 5.4. *The norm* $\|\mathscr{R}_0^{(m)\wedge}(t)\|$ *is an integrable function of* t, *its integral being bounded uniformly in* μ.

Proof. That properties (1) and (2) hold can be shown as in the proof of Lemma 3.1. The operator $\mathscr{R}_0^{(m)\wedge}(t)$ can be evaluated explicitly. It contains an exponential that decreases as t tends to $\pm\infty$. This makes it easy to complete the proof of the lemma.

THEOREM 5.7. *Let* $R^{(m)}(\lambda, \varphi)$ $(m = 2, 3,..., n-1)$ *be as in Lemma* 5.4. *Then the operator* $Q^{(n)}(\lambda, \varphi)$ *of Eq.* (1.10) *has the properties* (1)–(3) *of Lemma* 5.4. *The norm* $\|\mathscr{Q}^{(n)\wedge}(t)\|$ *is an integrable function of* t, *its integral being bounded uniformly in* μ.

Proof. By [3, Eqs. (6.33) and (6.38)], the operator $Q^{(n)}$ is a sum of operators

$$R_{p(l)}^{(n)}(\lambda, \varphi) = [(* R) * R_0^{(l)}]\,(\lambda, \varphi). \tag{5.14}$$

That properties (1)–(3) hold can therefore be shown as in the proof of Lemma 5.4. As in Eq. (5.7),

$$\mathscr{R}_{p(l)}^{(n)\wedge}(t) = (2\pi)^{-1/2}\,(* \mathscr{R})^{\wedge}\,(t)\,\mathscr{R}_0^{(l)\wedge}(t). \tag{5.15}$$

The norm of this quantity has the desired properties by Lemmas 5.4 and 5.6. Hence, so does the norm of $\mathscr{Q}^{(n)\wedge}(t)$.

6. THE FREDHOLM KERNEL

With reference to Eq. (1.10), it is convenient to define

$$F^{(n)}(\lambda, \varphi) = K^{(n)}(\lambda, \varphi)\, R^{(n)}(\lambda, \varphi). \tag{6.1}$$

The kernel $K^{(n)}(\lambda, \varphi)$ is a sum of kernels $K_{p(2)}^{(n)}$, by [3, Eq. (6.40)]. The subscript $p(2)$ runs through all possible ways of dividing n particles into groups of n_1 and n_2 particles, respectively, with $n_1 + n_2 = n$. If, in particular, $n_1 \geqslant 2$ and $n_2 \geqslant 2$, then

$$K_{p(2)}^{(n)} = -F^{(n_1)} * F^{(n_2)} * R_0^{(2)}[V - V_{p(2)}], \tag{6.2}$$

$V - V_{p(2)}$ denoting the interaction between the two groups. If $n_1 = 1$ and $n_2 = n - 1$, Eq. (6.2) must be replaced by

$$K_{p(2)}^{(n)} = -F^{(n-1)} * R_0^{(2)}[V - V_{p(2)}]. \tag{6.3}$$

It is the purpose of the present section to study the behavior of $K^{(n)}_{p(2)}$ as $|\lambda|$ tends to ∞. Under suitable assumptions, we find that $K^{(n)}_{p(2)}$ is in the Schmidt class and that its Schmidt norm tends to 0 if λ is in $\Gamma_0^{(n)}(\zeta, \varphi)$ and $|\lambda|$ tends to ∞. For large $|\lambda|$, it follows that the resolvent can be written as a Neumann series. This fact is used in the next section.

LEMMA 6.1. *Let $R^{(m)}(\lambda, \varphi)$ $(m = 2, 3,..., n - 1)$ be as in Lemma 5.4. Let $K^{(m)}(\lambda, \varphi)$ $(m = 2, 3,..., n - 1)$ have the following properties.*

(1) *It belongs to the Schmidt class on $\mathfrak{L}^2(3m - 3)$, its Schmidt norm being bounded uniformly in λ.*

(2) *There exists a constant $d^{(m)}$ such that*

$$\int_{-\infty}^{\infty} \{\sigma[\mathscr{K}^{(m)}(l)]\}^2\, dl < d^{(m)}, \tag{6.4}$$

uniformly in μ.

(3) *The operator $\mathscr{K}^{(m)\wedge}(t)$ belongs to the Schmidt class, its Schmidt norm being bounded uniformly in t and μ.*

(4) *The Schmidt norm $\sigma[\mathscr{K}^{(m)\wedge}(t)]$ is an integrable function of t, its integral being bounded uniformly in μ.*

Let $F^{(m)}(\lambda, \varphi)$ be defined by Eq. (6.1). *Then $F^{(m)}(\lambda, \varphi)$ has the properties* (1)–(3).

Proof. Properties (1) and (2) are obvious. Since K and K^* have the same Schmidt norm, Lemma 4.6 applies and Corollary 4.7 says that

$$\mathscr{F}^{(m)\wedge}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathscr{K}^{(m)\wedge}(t-u)\, \mathscr{R}^{(m)\wedge}(u)\, du. \tag{6.5}$$

Property (3) of $\sigma[\mathscr{F}^{(m)\wedge}(t)]$ now follows from property (4) of $\sigma[\mathscr{K}^{(m)\wedge}(t)]$ and property (3) of $\|\mathscr{R}^{(m)\wedge}(t)\|$.

LEMMA 6.2. *If $V(\varphi)$ belongs to the Schmidt class on $\mathfrak{L}^3$, then the operator*

$$K^{(2)}(\lambda, \varphi) = -R_0^{(2)}(\lambda, \varphi)\, V(\varphi) \tag{6.6}$$

has the properties of Lemma 6.1.

Proof. This is related to Lemmas 3.1 and 5.6, and is easily checked by explicit calculation.

LEMMA 6.3. *Let the data be as in Lemma* 6.1 *and let $V(\varphi)$ be a sum of two-particle interactions as in Section* 2. *Then the operators $K^{(n)}_{p(2)}(\lambda, \varphi)$ have the properties of Lemma* 6.1.

Proof. We first consider the case $n_1 = 1$, $n_2 = n - 1$ and refer to Eq. (6.3). By Corollary 4.7,

$$[\mathscr{F}^{(n-1)} * \mathscr{R}_0^{(2)}]^\wedge(t) = (2\pi)^{-1/2}\, \mathscr{F}^{(n-1)\wedge}(t)\, \mathscr{R}_0^{(2)\wedge}(t). \tag{6.7}$$

The operator $\mathscr{F}^{(n-1)}$ acts on the internal coordinates of the group of $n - 1$ particles, the operator $\mathscr{R}_0^{(2)}$ on the coordinates of the remaining particle relative to the center of mass of the large group. Let the latter coordinates be denoted by k_1, ω_1. The interaction $V - V_{p(2)}$ may then be regarded as an operator in the Schmidt class on the $\mathfrak{L}^2$-space of functions $f(k_1, \omega_1)$. According to this point of view, the internal coordinates of the $n - 1$ particles are parameters. The Schmidt norm of $V - V_{p(2)}$ is bounded uniformly in these. The operator $\mathscr{R}_0^{(2)}[V - V_{p(2)}]$ is in the Schmidt class on the space of functions $f(k_1, \omega_1)$ in the same sense. As such, it has all the properties of Lemma 6.1. This follows from Lemma 6.2.

In an obvious notation, we may now write

$$\sigma[\mathscr{K}_{p(2)}^{(n)\wedge}(t)] \leqslant (2\pi)^{-1/2}\, \sigma[\mathscr{F}^{(n-1)\wedge}(t)]\, \sigma[\mathscr{R}_0^{(2)\wedge}(t)\,(V - V_{p(2)})]. \tag{6.8}$$

This shows that properties (3) and (4) are satisfied and that the Schmidt norm on the left is a square-integrable function of t. Property (2) now follows from Lemma 4.3, property (1) from Lemma 4.4. This completes the argument for $n_1 = 1$, $n_2 = n - 1$. The case $n_1 \geqslant 2$, $n_2 \geqslant 2$ can be discussed in a similar way.

Remark 6.4. Suppose $R^{(m)}(\lambda, \varphi)$ satisfies the data of Lemma 5.4. Then so does $R^{(m)}(\bar{\lambda}, -\varphi)$, owing to Lemma 4.5 and Remark 5.5. Thus, if $K^{(m)}(\bar{\lambda}, -\varphi)$ has properties (1)–(4) of Lemma 6.1, then $F^{(m)}(\bar{\lambda}, -\varphi)$ has properties (1)–(3), and $K^{(n)}(\bar{\lambda}, -\varphi)$ has properties (1)–(4).

THEOREM 6.5. *Let the data be as in Lemma* 6.3. *Then* $K^{(n)}(\lambda, \varphi)$ *has the properties of Lemma* 6.1. *The Schmidt norm* $\sigma[K^{(n)}(\lambda, \varphi)]$ *tends to* 0 *as* $|\lambda|$ *tends to* ∞, *uniformly in* λ.

Proof. Since $K^{(n)}(\lambda, \varphi)$ is a sum of kernels $K_{p(2)}^{(n)}$, properties (1)–(4) follow from Lemma 6.3. The Schmidt norm tends to 0 as $|\lambda|$ tends to ∞ owing to property (4) and Lemma 4.4. It thus remains to show that the limit is uniform in λ, provided λ is in the region $\Gamma_0^{(n)}(\zeta, \varphi)$.

Let the kernel of $K^{(n)}(\lambda, \varphi)$ be denoted by $K(r, r', \lambda)$. Then

$$\sigma[K^{(n)}(\lambda, \varphi)] = \sup_{f \in \mathfrak{L}^2} \left| \|f\|^{-1} \int K(r, r', \lambda)\, f(r, r')\, dr\, dr' \right|. \tag{6.9}$$

Now divide the region $\Gamma_0^{(n)}(\zeta, \varphi)$ into a finite number of strips, plus two half-planes on either side of the set of strips. For example, suppose that there is a strip

$$\lambda = \mu + le^{2i\varphi}, \qquad \mu_1 \leqslant \mu \leqslant \mu_2, \quad 0 \leqslant l < \infty. \tag{6.10}$$

Consider the function

$$M(\lambda) = (\lambda - \mu_1)(\lambda - \mu_1 + me^{2i\varphi})^{-1} \| f \|^{-1} \int K(r, r', \lambda) f(r, r')\, dr\, dr', \tag{6.11}$$

with $m > 0$. Since $| \varphi | < \pi/2$ by assumption,

$$| M(\lambda)| \leqslant (\cos 2\varphi)^{-1/2} (l + \mu - \mu_1)(l + m)^{-1} \sigma[K^{(n)}(\lambda, \varphi)]. \tag{6.12}$$

Choosing some $\epsilon > 0$, we can find L such that $| M(\lambda)| < \epsilon$ if $l > L$ and λ is on the half-lines $\lambda = \mu_j + le^{2i\varphi}$ $(j = 1, 2)$. This is due to the Schmidt norm in Eq. (6.12) tending to 0 as $| \lambda |$ tends to ∞. Now take $l \leqslant L$. Since the Schmidt norm is bounded uniformly in λ, we can make $| M(\lambda)| < \epsilon$ in the region $0 \leqslant l \leqslant L$, $\mu_1 \leqslant \mu \leqslant \mu_2$ by making m sufficiently large. This makes $| M(\lambda)| < \epsilon$ everywhere on the boundary of the strip. It now follows from the Phragmén–Lindelöf theorem [19, Section 5.61] that $| M(\lambda)| < \epsilon$ everywhere inside the strip.

Owing to Eq. (6.11),

$$\begin{aligned} &\| f \|^{-1} \left| \int K(r, r', \lambda) f(r, r')\, dr\, dr' \right| \\ &\qquad \leqslant (\cos 2\varphi)^{-1/2} (l + m + \mu - \mu_1)\, l^{-1}\, | M(\lambda)|\,. \end{aligned} \tag{6.13}$$

This quantity is less than $2\epsilon(\cos 2\varphi)^{-1/2}$ if l exceeds some L' that is to be chosen such that $L' + m + \mu - \mu_1 \leqslant 2L'$. The point is that L' does not depend on f. It is fairly easy to show that $(K^{(n)}(\lambda, \varphi) f, g)$ has a uniform limit for any fixed f, g. Because of Eq. (6.9), the above shows that the limit of the Schmidt norm is also uniform, provided λ tends to ∞ through the strip (6.10). Other parts of the region $\Gamma_0^{(n)}(\zeta, \varphi)$ can be treated in the same way. This completes the proof of Theorem 6.5.

7. The Resolvent

The results of Sections 5 and 6 enable Lemma 3.4 to be generalized to any number of particles.

Lemma 7.1. *Let* $R^{(m)}(\lambda, \varphi)$ $(m = 2, 3, \ldots, n - 1)$ *be as in Lemma* 5.4, $K^{(m)}(\lambda, \varphi)$ *and* $K^{(m)}(\bar{\lambda}, -\varphi)$ $(m = 2, 3, \ldots, n - 1)$ *as in Lemma* 6.1. *Let* $V(\varphi)$ *be as in Section* 2. *Then* $R^{(n)}(\lambda, \varphi)$ *has the properties of Lemma* 5.4.

Proof. Let $R_L^{(n)}(\lambda, \varphi)$ be defined as in Eq. (3.11). It is obvious that it has the desired properties. Given ϵ, Theorem 6.5 says that L can be chosen so large that $\sigma[K^{(n)}(\lambda, \varphi)] < \epsilon$ whenever λ is in $\Gamma_0^{(n)}(\zeta, \varphi)$ and $|\lambda| > L$. Writing

$$R' = R^{(n)}(\lambda, \varphi) - R_L^{(n)}(\lambda, \varphi), \tag{7.1}$$

and similarly for Q' and K', we have

$$R' = Q' + K'R', \qquad R' = \sum_{p=0}^{\infty} (K')^p Q'. \tag{7.2}$$

Since Q' has property (1) of Lemma 5.4, so does R'. Also,

$$\| \mathscr{R}'(l) f \| \leqslant \sum_{p=0}^{\infty} \| \mathscr{K}'(l) \|^p \| \mathscr{Q}'(l) f \| \leqslant (1 - \epsilon)^{-1} \| \mathscr{Q}'(l) f \| . \tag{7.3}$$

Property (2) for R' therefore follows from property (2) of Q'.

Writing

$$\mathscr{R}'^{\wedge}(t) = \mathscr{Q}'^{\wedge}(t) + [\mathscr{K}'\mathscr{Q}']^{\wedge}(t) + \sum_{p=2}^{\infty} [(\mathscr{K}')^p \mathscr{Q}']^{\wedge}(t), \tag{7.4}$$

we proceed to show that $\| \mathscr{R}'^{\wedge}(t) \|$ is bounded uniformly in t. The norm of the first term on the right has the desired property by Theorem 5.7. By Corollary 4.7, the second term is equal to

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathscr{K}'^{\wedge}(t - u)\, \mathscr{Q}'^{\wedge}(u)\, du. \tag{7.5}$$

To take care of this, we observe that $\| \mathscr{Q}'^{\wedge}(u) \|$ is bounded and that $\| \mathscr{K}'^{\wedge}(t - u) \|$ has an integral with respect to u that is bounded uniformly in t, by property (4) of Lemma 6.1. As for the third term on the right in Eq. (7.4),

$$\sigma \left[\sum_{p=2}^{\infty} \{\mathscr{K}'(l)\}^p \mathscr{Q}'(l) \right] \leqslant \sum_{p=0}^{\infty} \| \mathscr{K}'(l) \|^p \{\sigma[\mathscr{K}'(l)]\}^2 \| \mathscr{Q}'(l) \| . \tag{7.6}$$

By property (2) of Lemma 6.1, the Schmidt norm on the right is square-integrable. Hence, the expression (7.6) is an integrable function of l. The desired property of the transformed quantity now follows from Lemma 4.4.

The foregoing may be summarized by saying that both $R_L^{(n)}$ and $R' = R^{(n)} - R_L^{(n)}$ have the properties (1)–(3) of Lemma 5.4. Hence, so does $R^{(n)}$. The data allow the adjoint $R^{(n)*}$ to be discussed in the same way. This completes the proof of Lemma 7.1.

THEOREM 7.2. *If the interaction $V(\varphi)$ is as in Section 2, then $R^{(n)}(\lambda, \varphi)$ has* *the properties of Lemma* 5.4, $Q^{(n)}(\lambda, \varphi)$ *has the properties of Theorem* 5.7, *and* $K^{(n)}(\lambda, \varphi)$ *has the properties of Lemma* 6.1 *and Theorem* 6.5.

Proof. This follows by induction. We start with Lemmas 5.6 and 6.2 for $R_0^{(2)}$ and $K^{(2)}$. Lemma 7.1 then takes care of $R^{(2)}$. Next, we proceed to Theorem 5.7 for $Q^{(3)}$ and Theorem 6.5 for $K^{(3)}$. Lemma 7.1 then takes care of $R^{(3)}$. And so on.

Remark 7.3. If f is any fixed element of $\mathfrak{L}^2$, the norm $\| R^{(n)}(\lambda, \varphi) f \|$ tends to 0 uniformly in λ if λ is in $\Gamma^{(n)}(\zeta, \varphi)$ and $| \lambda |$ tends to ∞. This follows from property (1) of $R^{(n)}(\lambda, \varphi)$ and Remark 3.5.

Remark 7.4. Since $R^{(n)}(\lambda, \varphi)$ can be evaluated with the help of a Neumann series if λ is in $\Gamma_0^{(n)}(\zeta, \varphi)$ and $| \lambda |$ is sufficiently large, there is at most a finite number of poles in $\Gamma_0^{(n)}(\zeta, \varphi)$. We cannot exclude the possibility of poles clustering around half-lines $Y(\lambda_p, \varphi)$. Specifically, they may cluster around λ_p. In an obvious terminology, the singularities of $R^{(n)}(\lambda, \varphi)$ thus fall into three categories, namely isolated poles, isolated branch cuts, and branch cuts of accumulation. Now suppose that the number of elements in each category is finite in case $m = 2, 3, \ldots, n - 1$. Because of the way the branch cuts $Y(\lambda_p, \varphi)$ are caused by convolutions, as discussed in Section 5, it follows that $Q^{(n)}(\lambda, \varphi)$ and $K^{(n)}(\lambda, \varphi)$ are regular except for a finite number of isolated branch cuts, plus a finite number of branch cuts of accumulation. Hence, $R^{(n)}(\lambda, \varphi)$ is regular except for a finite number of singularities in the three categories. Since $Q^{(2)}(\lambda, \varphi)$ and $K^{(2)}(\lambda, \varphi)$ have one isolated branch cut only, it follows by induction that the number of elements in each category is finite for every n.

To see what the above means, suppose that λ_p and λ_{p+1} are successive branch points on the real axis. Consider the strip between the half-lines $Y(\lambda_p + \eta, \varphi)$ and $Y(\lambda_{p+1} - \eta, \varphi)$, with some small $\eta > 0$. If we travel through the strip in the direction of $\infty e^{2i\varphi}$, we may encounter a finite number of branch points off the real axis at most. In other words, we need not be afraid that there is an infinite sequence $\{\mu_r\}$ everywhere dense in the interval $\lambda_p + \eta \leqslant \mu \leqslant \lambda_{p+1} - \eta$ and corresponding to a sequence of branch points $\{\mu_r + l_r e^{2i\varphi}\}$ which would prevent us from reaching the point at infinity. The strip splits into a finite number of substrips at most.

The above argument essentially refers to the imaginary parts of branch points λ_p being confined to some bounded interval. The real parts are likewise confined to a bounded interval. Hence, for any small $\zeta > 0$, the region $\Gamma_0^{(n)}(\zeta, \varphi)$ consists of the whole plane except for a finite number of half-strips making an angle 2φ with the positive real axis. This remark supplements qualitative statements in Section 5.

In Section 9, it is of considerable interest that we be able to compare resolvents $R^{(n)}(\lambda, \varphi)$ at different values of φ. The issue is the following. Suppose $Y(\lambda_p, \varphi)$ and $Y(\lambda_{p+1}, \varphi)$ are successive branch cuts and consider the half-lines $Y(\lambda_p + \eta, \varphi)$ and $Y(\lambda_{p+1} - \eta, \varphi)$, with some $\eta > 0$. To fix ideas, suppose that $\varphi > 0$ and that $Y(\lambda_p + \eta, \varphi)$ is to the left of $Y(\lambda_{p+1} - \eta, \varphi)$. If φ is allowed to vary, the half-lines will rotate. In doing so, they may pass through resonance poles or through branch points off the real axis. Suppose, however, that this does not occur if $\alpha \leqslant \varphi \leqslant \beta$, the angles α and β depending on η and on the branch points λ_p and λ_{p+1} that are being considered. Now choose f and g in $\mathfrak{G}(\alpha, \beta)$ and examine the integral

$$I(\lambda) = \int d\omega \int_0^\infty g(ke^{i\varphi}, \omega)\, R^{(n)}(\lambda, \varphi) f(ke^{i\varphi}, \omega)\, (ke^{i\varphi})^{3n-4}\, e^{i\varphi}\, dk. \tag{7.7}$$

If λ is in the resolvent set of H, then $I(\lambda)$ does not depend on φ by [1, Corollary 2.8]. It can be continued analytically in λ across the real axis into the strip between $Y(\lambda_p + \eta, \varphi)$ and $Y(\lambda_{p+1} - \eta, \varphi)$, then across $Y(\lambda_p + \eta, \varphi)$ and $Y(\lambda_{p+1} - \eta, \varphi)$. If it is understood that

$$0 < \alpha \leqslant \varphi \leqslant \beta < \pi/2, \tag{7.8}$$

then the funtion $I(\lambda)$ has a branch defined in the region between $Y(\lambda_{p+1} - \eta, \alpha)$ and $Y(\lambda_p + \eta, \beta)$. Let this region be denoted by Δ. If λ is any particular point in Δ, then $I(\lambda)$ can be evaluated by taking φ in Eq. (7.7) so that λ is between $Y(\lambda_p + \eta, \varphi)$ and $Y(\lambda_{p+1} - \eta, \varphi)$. Thus, there is some freedom in the choice of φ, yet we do need different values of φ as we travel through Δ. Now, if λ tends to ∞ along any particular path in Δ, then $I(\lambda)$ tends to 0 by Remark 7.3. It is hard to see, however, whether the limit is uniform in the angle. We wish to prove that this is indeed the case.

DEFINITION 7.5. If ζ, α, and β are fixed and there is a constant M such that

$$\| R^{(n)}(\lambda, \varphi)\| < M, \tag{7.9}$$

no matter how one first chooses φ in $[\alpha, \beta]$, next λ in $\Gamma^{(n)}(\zeta, \varphi)$, then $R^{(n)}(\lambda, \varphi)$ is said to be bounded uniformly in λ and φ.

In a similar way, various other properties considered in Sections 5 and 6 can be reformulated so as to include uniformity in φ. Suppose, therefore, that the data of Lemma 5.4 hold uniformly in φ. Then so does the proposition. It is true that $R^{(m)}(\lambda, \varphi)$ may have different numbers of branch cuts, and poles off the real axis, for different values of φ. Even so, no matter how we choose φ, the quantities considered in Lemma 5.4 can be evaluated by adding contributions from a finite number of convolution integrals along straight lines. Since the contribution from each integral is bounded uniformly in φ by the present assumption, so is the total result.

It is easy to check that Lemmas 5.6 and 6.2 hold uniformly in φ. So do Theorem 5.7 and Lemmas 6.1, 6.3, provided the data of Lemmas 5.4 and 6.1 hold uniformly in φ. If this is so, then the proof of Theorem 6.5 can be adapted to show that the Schmidt norm $\sigma[K^{(n)}(\lambda, \varphi)]$ tends to 0 uniformly in λ and φ. This means that, given $\epsilon > 0$, there exists an L such that

$$\sigma[K^{(n)}(\lambda, \varphi)] < \epsilon \tag{7.10}$$

whenever $|\lambda| > L$, no matter how we choose φ in $[\alpha, \beta]$, then λ in $\Gamma_0^{(n)}(\zeta, \varphi)$.

LEMMA 7.6. *If the data of Lemma* 7.1 *hold uniformly in* φ, *then* $R^{(n)}(\lambda, \varphi)$ *has the properties of Lemma* 5.4 *uniformly in* φ.

Proof. The proof of Theorem 7.1 can easily be adapted to show that the operator R' of Eq. (7.1) has the desired properties. As for $R_L^{(n)}(\lambda, \varphi)$, if this is bounded uniformly in λ and φ, then it is integrable and square-integrable with respect to l, uniformly in φ. Hence, it also has the desired properties. This then proves the lemma.

The above indicates that it only remains to study the norm of $R_L^{(n)}(\lambda, \varphi)$. Referring to the Fredholm equation (1.10), we write

$$R_L^{(n)}(\lambda, \varphi) = Q_L^{(n)}(\lambda, \varphi) + \Delta(\lambda, \varphi)\, Q_L^{(n)}(\lambda, \varphi)/\delta(\lambda), \tag{7.11}$$

as in [3], Eq. (6.51). For any fixed φ, the quantity $\Delta(\lambda, \varphi)$ is an operator in the Schmidt class on $\mathfrak{L}^2$. Its Schmidt norm is bounded uniformly in λ and φ, owing to the fact that $\sigma[K^{(n)}(\lambda, \varphi)]$ is bounded uniformly in λ and φ. For any fixed λ, the quantity $\delta(\lambda)$ is a number, likewise bounded uniformly in λ and φ. In fact, if we first choose φ, then $|\lambda| < L$ and λ in $\Gamma_0^{(n)}(\lambda, \varphi)$, then there is a small interval $[\varphi_1, \varphi_2]$ such that λ is in $\Gamma_0^{(n)}(\lambda, \varphi)$ for all φ in $[\varphi_1, \varphi_2]$. At the chosen λ, we may therefore evaluate $\delta(\lambda)$ for all φ in $[\varphi_1, \varphi_2]$. It does not depend on φ by [1, Corollary 5.2].

If φ now runs from α to β, then $\Gamma_0^{(n)}(\zeta, \varphi)$ covers a multisheeted region in the λ-plane. Let this be denoted by $\Gamma_0^{(n)}(\zeta)$. The function $\delta(\lambda)$ can be continued analytically to $\Gamma_0^{(n)}(\zeta)$. To do this, it suffices to choose λ in $\Gamma_0^{(n)}(\zeta)$, next φ in such a way that λ is in $\Gamma_0^{(n)}(\zeta, \varphi)$. One can then evaluate $\delta(\lambda)$ with the help of [2, Eq. (4.6)]. Since this procedure involves quantities that are all bounded uniformly in λ and φ, it follows that $\delta(\lambda)$ is bounded uniformly in λ provided λ is in $\Gamma_0^{(n)}(\zeta)$.

In the intersection of $\Gamma_0^{(n)}(\zeta)$ and $|\lambda| < L$, the function $\delta(\lambda)$ may have a finite number of zeros. Some of these may be poles of the resolvent. We delete ζ-neighborhoods of these form $\Gamma_0^{(n)}(\zeta)$ to obtain $\Gamma^{(n)}(\zeta)$, the lemma only referring to the region in the λ-plane that is at least a distance ζ away from the spectrum of $H(\varphi)$.

It is well known that $\delta(\lambda)$ may also have spurious zeros [20, 21], that is, zeros in the resolvent set of $H(\varphi)$. If $\Gamma^{(n)\prime}(\zeta)$ is the region obtained from $\Gamma^{(n)}(\zeta)$ by deleting ϵ-neighborhoods of any spurious zeros, then $R_L^{(n)}(\lambda, \varphi)$ is bounded uniformly in λ and φ for all φ in $[\alpha, \beta]$ and λ in the intersection of $\Gamma^{(n)}(\zeta, \varphi)$ and $\Gamma^{(n)\prime}(\zeta)$. This follows from Eq. (7.11) and the foregoing results concerning $\Delta(\lambda, \varphi)$ and $\delta(\lambda)$.

It remains to examine $R_L^{(n)}(\lambda, \varphi)$ in ϵ-neighborhoods of spurious zeros in $\Gamma^{(n)}(\zeta, \varphi)$. Let $\lambda = \lambda_s$ be a spurious zero of order t. Then $(\lambda - \lambda_s)^t/\delta(\lambda)$ is bounded in a sufficiently small ϵ-neighborhood of λ_s. Since $\Delta(\lambda, \varphi)$ is analytic in λ, it may be expanded in a Taylor series in $\lambda - \lambda_s$. The coefficients can be written as integrals involving $\Delta(\lambda, \varphi)$ along the circle $|\lambda - \lambda_s| = \epsilon$. Since $\Delta(\lambda, \varphi)$ is bounded uniformly in λ and φ, the Taylor series converges uniformly in λ and φ, in the uniform operator topology on $\mathfrak{L}^2$. This follows from Cauchy's inequality for analytic functions on a disc [19, Section 2.5]. If the zero is to be spurious, then the first few terms of the Taylor series must vanish identically, up to the term with $(\lambda - \lambda_s)^{t-1}$. The remainder is $(\lambda - \lambda_s)^t$ times an operator that is bounded uniformly in λ and φ. If the remainder is divided by $\delta(\lambda)$, this yields an operator that is likewise bounded uniformly in λ and φ, it being understood that $|\lambda - \lambda_s| < \epsilon$. Since all spurious zeros can be taken care of in this way, it follows that $R_L^{(n)}(\lambda, \varphi)$ is bounded uniformly in λ and φ. This suffices to prove the theorem.

COROLLARY 7.7. *If the interaction $V(\varphi)$ is as in Sec. 2, then the proposition of Theorem 7.2 holds uniformly in φ. In particular, $\| R^{(n)}(\lambda, \varphi)\|$ is bounded uniformly in λ and φ.*

Proof. This follows from the same induction argument as Theorem 7.2.

THEOREM 7.8. *Suppose that*

$$0 < \alpha \leqslant \varphi \leqslant \beta < \pi/2, \qquad \lambda_p < \lambda_{p+1}, \qquad \eta > 0. \tag{7.12}$$

Let $Y(\lambda_p, \varphi)$ and $Y(\lambda_{p+1}, \varphi)$ be successive branch cuts of $R^{(n)}(\lambda, \varphi)$ and suppose that $Y(\lambda_p + \eta, \varphi)$ is to the left of $Y(\lambda_{p+1} - \eta, \varphi)$. Let Δ be the region bounded by $Y(\lambda_{p+1} - \eta, \alpha)$ and $Y(\lambda_p + \eta, \beta)$, plus a curve between $\lambda_p + \eta$ and $\lambda_{p+1} - \eta$. Define the function $I(\lambda)$ by Eq. (7.7) and suppose that it has neither poles nor branch points in Δ. Then $I(\lambda)$ tends to 0 as $|\lambda|$ tends to ∞, uniformly in the angle. A similar result applies in case $-\pi/2 < \alpha \leqslant \varphi \leqslant \beta < 0$.

Proof. It was explained above that $I(\lambda)$ is analytic in Δ and tends to 0 as λ

tends to ∞ along $\Upsilon(\lambda_{p+1} - \eta, \alpha)$ or $\Upsilon(\lambda_p - \eta, \beta)$. Now take any λ in Δ and choose an appropriate φ in Eq. (7.7). This yields

$$|I(\lambda)| \leqslant \| R^{(n)}(\lambda, \varphi)\| \, \| f(ke^{i\varphi}, \omega)\| \, \| g(ke^{i\varphi}, \omega)\| . \tag{7.13}$$

The first factor on the right is bounded uniformly in λ and φ by Corollary 7.7. The other factors are bounded uniformly in φ owing to [1, Eq. (2.36)] or [2, Eq. (2.31)]. Hence, $|I(\lambda)|$ is bounded uniformly in λ. The Phragmén–Lindelöf theorem [19, Section 5.63] now completes the proof. The case $-\pi/2 < \alpha < \beta < 0$ can be discussed in a similar way.

8. Projection Operators

The present section applies to any number of particles. We therefore use $R(\lambda, \varphi)$ to indicate $R^{(n)}(\lambda, \varphi)$ with any n.

Suppose that the half-line $\Upsilon(\lambda_p, \varphi)$ is a branch cut of $R(\lambda, \varphi)$ and consider the oriented contour C_p consisting of the lines

$$\begin{aligned} \lambda &= \lambda_p + (l - i\epsilon)\, e^{2i\varphi} \qquad (\infty > l > -\infty), \\ \lambda &= \lambda_p + (l + i\epsilon)\, e^{2i\varphi} \qquad (-\infty < l < \infty). \end{aligned} \tag{8.1}$$

The branch cut $\Upsilon(\lambda_p, \varphi)$ is to the right of C_p, the region to the left of C_p consisting of the two half-planes

$$| \operatorname{Im}(\lambda - \lambda_p)\, e^{-2i\varphi} | > \epsilon. \tag{8.2}$$

Let C_{pL} be the part of C_p on which $| l | < L$, so that C_{pL} tends to C_p as L tends to ∞. Let ϵ be such that $R(\lambda, \varphi)$ has no singularities on C_p. This geometry allows the following theorem.

Theorem 8.1. *Let the contours C_p and C_{pL} be as above. Then the limit*

$$(2\pi i)^{-1} \lim_{L\to\infty} \int_{C_{pL}} (R(\lambda, \varphi) f, g)\, d\lambda \tag{8.3}$$

exists and can be written in the form $(P_p(\varphi) f, g)$, the operator $P_p(\varphi)$ having the following properties.

(1) *It is a bounded linear operator on $\mathfrak{L}^2$.*

(2) *If ϵ in Eq. (8.1) is varied continuously subject to the condition that C_p does not pass through any singularities of $R(\lambda, \varphi)$, then $P_p(\varphi)$ remains unchanged.*

(3) *It is idempotent,*

$$P_p(\varphi)\,P_p(\varphi) = P_p(\varphi). \tag{8.4}$$

(4) *If λ is in the resolvent set of $H(\varphi)$, then*

$$P_p(\varphi)\,R(\lambda, \varphi) = R(\lambda, \varphi)\,P_p(\varphi). \tag{8.5}$$

(5) *The operator* (8.5) *can be continued analytically in λ to the region to the left of C_p by means of the relation*

$$(P_p(\varphi)\,R(\lambda, \varphi) f, g) = (2\pi i)^{-1} \lim_{L\to\infty} \int_{C_{pL}} (P_p(\varphi)\,R(\mu, \varphi) f, g)\,(\mu - \lambda)^{-1}\,d\mu. \tag{8.6}$$

(6) *If $P_q(\varphi)$ is determined by a contour C_q to the left of C_p, running through one of the half-planes* (8.2), *then*

$$P_p(\varphi)\,P_q(\varphi) = P_q(\varphi)\,P_p(\varphi) = 0. \tag{8.7}$$

(7) *If ϵ is so large that $R(\lambda, \varphi)$ is regular in the half-planes* (8.2), *then $P_p(\varphi)$ is the identity operator.*

Proof. The limit (8.3) exists because $R(\lambda, \varphi)$ has property (2) of Lemma 5.4, by Theorem 7.2. This can be shown as in Section 3. Specifically, the quantity (8.3) is less than some constant times $\|f\|\,\|g\|$. Hence, it defines a bounded operator on $\mathfrak{L}^2$, which is denoted by $P_p(\varphi)$.

Since $R(\lambda, \varphi)$ is regular and $\|R(\lambda, \varphi) f\|$ tends to 0 as l tends to $\pm\infty$, uniformly in ϵ under the variations envisaged in proposition (2) of the lemma, the limit (8.3) does not depend on ϵ. Hence, neither does $P_p(\varphi)$, provided the variations in ϵ are sufficiently small.

The above shows that $P_p(\varphi)$ has properties (1) and (2). To prove property (3), it now suffices to show that

$$P_p(\varphi)\,P_p(\varphi) f = P_p(\varphi) f \tag{8.8}$$

is true for all f in a dense set in $\mathfrak{L}^2$. Let this set be the domain of $H^2(\varphi)$ and let λ_0 be some point in the region (8.2). Since

$$R(\lambda, \varphi) f = (\lambda_0 - \lambda)^{-1} - (\lambda_0 - \lambda)^{-1} R(\lambda, \varphi)\,[H(\varphi) - \lambda_0] f, \tag{8.9}$$

and since

$$\lim_{L\to\infty} \int_{C_{pL}} (\lambda_0 - \lambda)^{-1}\,d\lambda = 0, \tag{8.10}$$

we may write

$$(P_p(\varphi)f, g) = (2\pi i)^{-1} \int_{C_p} (R(\lambda, \varphi)\,[H(\varphi) - \lambda_0]f, g)\,(\lambda - \lambda_0)^{-1}\,d\lambda, \tag{8.11}$$

hence

$$(P_p(\varphi)\,P_p(\varphi)f, g) = (2\pi i)^{-2} \int_{C_p} d\lambda \int_{D_p} (R(\lambda, \varphi)\,R(\mu, \varphi)\,[H(\varphi) - \lambda_0]^2 f, g) \times (\lambda - \lambda_0)^{-1}\,(\mu - \mu_0)^{-1}\,d\mu. \tag{8.12}$$

The contour D_p must be such that $R(\lambda, \varphi)$ is regular between C_p and D_p. To be specific, let λ_0 be to the left, C_p to the right of D_p, so that C_p is between D_p and the branch cut $Y(\lambda_p, \varphi)$.

In the integrand in Eq. (8.12), we now use the equation

$$R(\lambda, \varphi)\,R(\mu, \varphi) = (\lambda - \mu)^{-1}\,[R(\lambda, \varphi) - R(\mu, \varphi)]. \tag{8.13}$$

This results in two terms whose integrals converge separately. In the second term, it is most convenient to integrate with respect to λ first. This gives 0, due to the location of C_p relative to D_p. Integrating the first term with respect to μ gives

$$(2\pi i)^{-1} \int_{C_p} (R(\lambda, \varphi)\,[H(\varphi) - \lambda_0]^2 f, g)\,(\lambda - \lambda_0)^{-2}\,d\lambda. \tag{8.14}$$

Now, referring to Eq. (8.9), we may actually write

$$\begin{aligned} R(\lambda, \varphi)f = (\lambda_0 - \lambda)^{-1} - (\lambda_0 - \lambda)^{-2}\,[H(\varphi) - \lambda_0]f \\ + (\lambda_0 - \lambda)^{-2}\,R(\lambda, \varphi)\,[H(\varphi) - \lambda_0]^2 f. \end{aligned} \tag{8.15}$$

Since $(\lambda_0 - \lambda)^{-2}$ is regular between the two half-lines of C_p, it follows that the expression (8.14) is equal to $(P_p(\varphi)f, g)$, as we wished to prove.

This establishes property (3). Property (6) can be proved in much the same way. Property (4) is obvious.

To prove property (5), we first take λ to the left of C_p in the resolvent set of $H(\varphi)$. With the help of Eq. (8.13), it is easy to arrive at the expression (8.6). The right side of this can now be continued in λ to the whole region to the left of C_p.

It remains to prove property (7). To this end, we write

$$(R(\lambda, \varphi)f, g) = (R_0(\lambda, \varphi)f, g) - (R_0(\lambda, \varphi)\,V(\varphi)\,R(\lambda, \varphi)f, g) \tag{8.16}$$

and choose both f and g in the domain of $H(\varphi)$. Since $Y(0, \varphi)$ is the only singularity of $R_0(\lambda, \varphi)$ and since this is also a singularity of $R(\lambda, \varphi)$, the data imply that the second term on the right is regular in the half-planes (8.2). By Eq. (8.9), it is of order $O(|\lambda|^{-2})$ if λ is the region (8.2) and $|\lambda|$ tends to ∞. Hence, its integral along C_p vanishes. The first term can be integrated explicitly to give $2\pi i(f, g)$. In obtaining this result, it is assumed that f and g are in the domain of $H(\varphi)$. Since this is dense in $\mathfrak{L}^2$, the argument can easily be completed to prove property (7).

Remark 8.2. Let $\mathfrak{D}[H(\varphi)]$ be the domain of $H(\varphi)$ and let $P_p(\varphi)\,\mathfrak{L}^2$ be the range of $P_p(\varphi)$. Owing to property (4) of Theorem 8.1, $P_p(\varphi)$ maps $\mathfrak{D}[H(\varphi)]$ into $\mathfrak{D}[H(\varphi)]$ and satisfies

$$P_p(\varphi)\,H(\varphi)\,f = H(\varphi)\,P_p(\varphi)\,f \tag{8.17}$$

for all f in $\mathfrak{D}[H(\varphi)]$. Hence, $P_p(\varphi)\,\mathfrak{L}^2$ is an invariant subspace of $H(\varphi)$. Property (5) means that the spectrum of $P_p(\varphi)\,H(\varphi)$ is confined to the strip to the right of C_p. This strip is in the resolvent set of $[I - P_p(\varphi)]\,H(\varphi)$.

Remark 8.3. In straightforward cases, the number of branch cuts $Y(\lambda_p, \varphi)$ is finite, so there are no branch cuts of accumulation. One can then choose a set of contours C_p so that each one has only one branch cut on the right, the rest of the spectrum of $H(\varphi)$ being on the left. Proposition (7) of Theorem 8.1 clearly indicates that our definition allows more than one branch cut to be to the right of C_p. In other words, the subscript p does not specify $P_p(\varphi)$ completely. If branch cuts cluster around some particular $Y(\lambda_q, \varphi)$, then we cannot do better than to draw C_q so that the whole cluster is to its right.

Remark 8.4. Since $R(\lambda, \varphi)$ is regular in the lower or the upper half-plane, according as φ is positive or negative, and goes to 0 sufficiently fast as $|\lambda|$ tends to ∞, the contour C_p may be replaced by two half-lines plus some finite path B_p running from $\lambda = \lambda_p - i\epsilon e^{2i\varphi}$ to $\lambda = \lambda_p + i\epsilon e^{2i\varphi}$. In fact, it may happen that λ_p is off the real axis and that there is a pole on the real axis precisely between the two lines (8.1). One will then want to consider an integration contour passing between λ_p and the pole on the real axis. There is no difficulty in showing that this also yields a projection.

9. Analytic Projection Operators

The present section combines Theorems 7.8 and 8.1. The angle φ is varied in an interval $[\alpha, \beta]$. This results in the lines (8.1) tracing out regions Δ_+ and Δ_- in the λ-plane. For any f, g in $\mathfrak{G}(\alpha, \beta)$, the integral $I(\lambda)$ defined in Eq. (7.7) has branches that are analytic in Δ_+ and Δ_-, respectively. It is assumed that α, β, ϵ are such that $I(\lambda)$ has neither poles nor branch points in Δ_- and Δ_+. This assumption determines what values of α, β, ϵ are allowed. We now choose f in Eq. (8.3) so that it is the restriction, to fixed φ, of a function $f(ke^{i\varphi}, \omega)$ in $\mathfrak{G}(\alpha, \beta)$. We wish to prove that this makes $P_p(\varphi) f$ the restriction of a function in $\mathfrak{G}(\alpha, \beta)$. In fact, it is stated in Theorem 9.2 that $P_p(\varphi)$ is associated with an operator P_p in the class $\mathfrak{A}$, this being a class of bounded operators on $\mathfrak{G}(\alpha, \beta)$ that was introduced in [1, Definition 4.1].

The proof is based on the fact that $f(ke^{i\varphi}, \omega)$ belongs to $\mathfrak{G}(\alpha, \beta)$ if and only if it is an inverse Mellin transform according to

$$f(ke^{i\varphi}, \omega)\,(ke^{i\varphi})^{(3n-4)/2} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathbf{f}(u, \omega)\,(ke^{i\varphi})^{-iu-1/2}\,du, \tag{9.1}$$

with some function $\mathbf{f}(u, \omega)$ satisfying

$$\int d\omega \int_{-\infty}^{\infty} (e^{2\alpha u} + e^{2\beta u})\,|\,\mathbf{f}(u, \omega)|^2\,du < \infty. \tag{9.2}$$

This follows from [1, Theorem 2.9]. If f and g belong to $\mathfrak{G}(\alpha, \beta)$, then it follows from [1, Corollary 2.8] that the integral

$$\int d\omega \int_0^{\infty} f(ke^{i\varphi}, \omega)\,g(ke^{i\varphi}, \omega)\,(ke^{i\varphi})^{3n-4}\,e^{i\varphi}\,dk, \tag{9.3}$$

regarded as a function of φ, is constant in $\alpha \leqslant \varphi \leqslant \beta$.

For our present purpose, it is convenient to supplement the above as follows.

LEMMA 9.1. *Let ω be a set of $3n-4$ real variables taking values in bounded intervals. Suppose that α and β are real and $\alpha < \beta$. Let $f(k, \omega, \varphi)$ be defined on $0 \leqslant k < \infty$, the appropriate region for ω, and $\alpha \leqslant \varphi \leqslant \beta$. Suppose that*

$$\int d\omega \int_0^{\infty} |\,f(k, \omega, \varphi)|^2\,k^{3n-4}\,dk < \infty \tag{9.4}$$

for $\alpha \leqslant \varphi \leqslant \beta$. Suppose that the integral

$$\int d\omega \int_0^{\infty} f(k, \omega, \varphi)\,g(ke^{i\varphi}, \omega)\,(ke^{i\varphi})^{3n-4}\,e^{i\varphi}\,dk \tag{9.5}$$

does not depend on φ, no matter how $g(ke^{i\varphi}, \omega)$ is chosen in $\mathfrak{G}(\alpha, \beta)$. Then $f(k, \omega, \varphi)$ belongs to $\mathfrak{G}(\alpha, \beta)$.

Proof. The data allow us to take the Mellin transform

$$\mathbf{f}(u, \omega, \varphi) = (2\pi)^{-1/2} \int_0^{\infty} f(k, \omega, \varphi)\,(ke^{i\varphi})^{(3n-4)/2}\,k^{iu-1/2}\,dk, \tag{9.6}$$

which satisfies the relation

$$\int d\omega \int_{-\infty}^{\infty} |\, \mathbf{f}(u, \omega, \varphi)|^2 \, du = \int d\omega \int_{0}^{\infty} |\, f(k, \omega, \varphi)|^2 \, k^{3n-4} \, dk. \tag{9.7}$$

It is implied by Eq. (9.1) that the Mellin transform of $g(ke^{i\varphi}, \omega)\,(ke^{i\varphi})^{(3n-4)/2}$ is of the form

$$\mathbf{g}(u, \omega, \varphi) = e^{\varphi u - i\varphi/2}\mathbf{g}(u, \omega), \tag{9.8}$$

where $\mathbf{g}(u, \omega)$ satisfies a relation like Eq. (9.2). By general formulas for Mellin transforms, the expression (9.5) is equal to

$$\int d\omega \int_{-\infty}^{\infty} e^{-\varphi u + i\varphi/2}\mathbf{f}(u, \omega, \varphi)\, \mathbf{g}(-u, \omega)\, du. \tag{9.9}$$

Since this does not depend on φ by assumption, no matter how $\mathbf{g}(-u, \omega)$ is chosen in a set that is dense in $\mathfrak{L}^2$, it follows that there must be a function $\mathbf{f}(u, \omega)$ such that

$$\mathbf{f}(u, \omega, \varphi) = e^{\varphi u - i\varphi/2}\mathbf{f}(u, \omega). \tag{9.10}$$

This satisfies Eq. (9.2) due to Eqs. (9.4) and (9.7). Taking the inverse Mellin transform of $\mathbf{f}(u, \omega, \varphi)$ and using Eq. (9.10) shows that $f(k, \omega, \varphi)\,(ke^{i\varphi})^{(3n-4)/2}$ is equal to the right side of Eq. (9.1). Hence, $f(k, \omega, \varphi)$ is a function $f(ke^{i\varphi}, \omega)$ in $\mathfrak{G}(\alpha, \beta)$, as we wished to prove.

The next theorem contains a reference to the Hamiltonian H with resolvent $R(\lambda)$ on $\mathfrak{G}(\alpha, \beta)$. These quantities are related to $H(\varphi)$ and $R(\lambda, \varphi)$ on $\mathfrak{L}^2$ as in Eq. (1.2), the relation applying for f in $\mathfrak{G}(\alpha, \beta)$. Details are explained in [2, Sections 3.1, 3.2] and [3, Section 6.1].

THEOREM 9.2. *Let Δ_+ and Δ_- be the regions in the λ-plane traced out by the lines* (8.1) *as φ runs from α to β. Suppose that either $0 < \alpha < \beta < \pi/2$ or $-\pi/2 < \alpha < \beta < 0$, and that α, β, ϵ are such that the integral $I(\lambda)$ defined by Eq.* (7.7) *has neither poles nor branch points in $\Delta_- \cup \Delta_+$. Define the operator $P_p(\varphi)$ as in Theorem* 8.1.

Then $P_p(\varphi) f(ke^{i\varphi}, \omega)$ belongs to $\mathfrak{G}(\alpha, \beta)$ whenever $f(ke^{i\varphi}, \omega)$ belongs to $\mathfrak{G}(\alpha, \beta)$. There is an operator P_p in the class $\mathfrak{A}$ on $\mathfrak{G}(\alpha, \beta)$ such that

$$P_p f(ke^{i\varphi}, \omega) = P_p(\varphi) f(ke^{i\varphi}, \omega), \tag{9.11}$$

for all $f(ke^{i\varphi}, \omega)$ in $\mathfrak{G}(\alpha, \beta)$.

The operator P_p is idempotent. It commutes with $R(\lambda)$ whenever λ is in the *resolvent set of H. If $P_q(\varphi)$ annihilates $P_p(\varphi)$ in the sense of Eq.* (8.7) *and allows the same α, β, so that there is also an operator P_q on $\mathfrak{G}(\alpha, \beta)$, then*

$$P_p P_q = P_q P_p = 0.$$

Proof. The following applies to the case $0 < \alpha < \beta < \pi/2$. The case $-\pi/2 < \alpha < \beta < 0$ can be discussed in the same way.

Choosing f and g in $\mathfrak{G}(\alpha, \beta)$, we want to show that the expression $Z(\varphi)$ defined by

$$Z(\varphi) = \int d\omega \int_0^\infty g(ke^{i\varphi}, \omega)\, P_p(\varphi) f(ke^{i\omega}, \varphi)\, (ke^{i\varphi})^{3n-4}\, e^{i\varphi}\, dk \tag{9.12}$$

does not depend on φ. This result then makes it possible to utilize Lemma 8.5. To obtain the quantity (9.12), the function $I(\lambda)$ must be integrated along a contour in the λ-plane, the difficulty being that this necessarily depends on φ. To solve this problem, we begin by assuming that f in $\mathfrak{G}(\alpha, \beta)$ belongs to the domain of H^2. This implies that its restriction to fixed φ belongs to the domain of $H^2(\varphi)$ in $\mathfrak{L}^2$. Hence, Eq. (8.15) applies, λ_0 being a point that may conveniently be chosen on the negative real axis close to $-\infty$.

If Eq. (8.15) is used in evaluating $P_p(\varphi)$, its first and second terms on the right do not contribute. Thus, $R(\lambda, \varphi) f$ may be replaced by

$$(\lambda_0 - \lambda)^{-2} R(\lambda, \varphi)\, [H(\varphi) - \lambda_0]^2 f, \tag{9.13}$$

as in Eq. (8.14). This modification changes $I(\lambda)$ into $J(\lambda)$, say.

The analytic properties of $J(\lambda)$ are much the same as those of $I(\lambda)$, except that $J(\lambda)$ is of order $O(|\lambda|^{-2})$ as $|\lambda|$ tends to ∞, uniformly in the angle by Corollary 7.7 and Theorem 7.8.

The quantity $Z(\varphi)$ is $1/2\pi i$ times the sum of the integrals of $J(\lambda)$ taken along the lines (8.1). Now refer to the way Theorem 8.1 exhibits $P_p(\varphi)$ as a limit with respect to the contour C_{pL}. In an obvious notation,

$$P_p(\varphi) = \lim_{L\to\infty} P_{pL}(\varphi), \qquad Z(\varphi) = \lim_{L\to\infty} Z_L(\varphi). \tag{9.14}$$

To be specific, for any $\delta > 0$ there is an L_1, not depending on φ, such that

$$|Z(\varphi) - Z_L(\varphi)| < \delta \tag{9.15}$$

for $L > L_1$ and all φ in $[\alpha, \beta]$. This is due to $\| R(\lambda, \varphi)\|$, $\| g \|$, and $\|[H(\varphi) - \lambda_0]^2 f\|$ being bounded uniformly in λ, φ, as in the proof of Theorem 7.8.

Next, consider $Z_L(\beta) - Z_L(\alpha)$. This is the sum of

$$(2\pi i)^{-1} \int_{a+}^{b+} J(\lambda)\, d\lambda - (2\pi i)^{-1} \int_{a-}^{b-} J(\lambda)\, d\lambda, \tag{9.16}$$

with

$$\begin{aligned} a\pm &= \lambda_p + (L \pm i\epsilon)\, e^{2i\alpha}, \\ b\pm &= \lambda_p + (L \pm i\epsilon)\, e^{2i\beta}, \end{aligned} \tag{9.17}$$

plus two similar integrals along arcs in the lower half-plane. Since $J(\lambda)$ is of order $O(|\lambda|^{-2})$ uniformly in the angle, as observed above, there is an L_2 such that

$$|Z_L(\beta) - Z_L(\alpha)| < \delta \tag{9.18}$$

for $L > L_2$. Combining Eqs. (9.15) and (9.18) shows that $Z(\beta)$ equals $Z(\alpha)$. The argument can be repeated to prove that $Z(\varphi)$ equals $Z(\alpha)$ for every φ in $[\alpha, \beta]$.

The above assumes that f is in the domain $\mathfrak{D}(H^2)$. To remove this restriction, we choose f in $\mathfrak{G}(\alpha, \beta)$ and approximate this, in the norm on $\mathfrak{G}(\alpha, \beta)$, by a sequence $\{f_n\}$ $(n = 1, 2,...)$ in $\mathfrak{D}(H^2)$. Owing to [1, Eq. (2.36)] and [2, Eq. (2.31)], the restriction of f to fixed φ is approximated, in the norm on $\mathfrak{L}^2$, by the sequence in $\mathfrak{D}[H^2(\varphi)]$ consisting of the restrictions of f_n to fixed φ. Now remember that $P_p(\varphi)$ is a bounded operator on $\mathfrak{L}^2$. This means that we can choose $\delta > 0$, then make n so large that

$$|Z_n(\varphi) - Z(\varphi)| < \delta \qquad (\varphi = \alpha, \beta), \tag{9.19}$$

$Z_n(\varphi)$ being the quantity obtained from $Z(\varphi)$ if f is replaced by f_n. Since it was shown above that $Z_n(\beta)$ equals $Z_n(\alpha)$, it follows that $Z(\beta)$ equals $Z(\alpha)$ no matter how f and g are chosen in $\mathfrak{G}(\alpha, \beta)$. So does $Z(\varphi)$ for all φ in $[\alpha, \beta]$.

Since $P_p(\varphi)$ is a bounded operator on $\mathfrak{L}^2$ by Theorem 8.1, it is obvious that $P_p(\varphi) f(ke^{i\varphi}, \omega)$ is in $\mathfrak{L}^2$, for $\alpha \leqslant \varphi \leqslant \beta$. Lemma 8.5 now says that $P_p(\varphi) f(ke^{i\varphi}, \omega)$ belongs to $\mathfrak{G}(\alpha, \beta)$. Thus, if P_p is defined by Eq. (9.11), it is clear that P_p is an operator on $\mathfrak{G}(\alpha, \beta)$. That it belongs to the class $\mathfrak{A}$, and therefore is bounded, follows from [1, Theorem 4.12]. That P_p is idempotent, commutes with $R(\lambda)$ and P_q, and annihilates P_q, is readily deduced from the corresponding properties (8.4–8.7) of $P_p(\varphi)$. This concludes the proof of Theorem 9.2.

Remark 9.3. Along the lines of Remark 8.2, it is not difficult to show that the range of P_p is an invariant subspace of H. If $Y(\lambda_p, \varphi)$ is an isolated branch cut and $R(\lambda, \varphi)$ has no other singularities between the lines (8.1), then the spectrum of $P_p(\varphi)\,H(\varphi)$ is precisely the half-line $\Upsilon(\lambda_p, \varphi)$. Hence, the sector

$$2\beta < \arg(\lambda - \lambda_p) < 2\pi + 2\alpha, \qquad 0 < |\lambda - \lambda_p| \tag{9.20}$$

belongs to the resolvent set of $P_p(\varphi)\,H(\varphi)$ for all φ in $[\alpha, \beta]$. It therefore belongs to the resolvent set of P_pH by [3, Theorem 6.4].

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